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HYDRODYNAMIC EFFECTS OF NUCLEAR EXPLOSIONS

VOLUME V: THEORETICAL DEVELOPMENTS

NATIONAL ENGINEERING SCIENCE CO.

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92

HYDRODYNAMIC EFFECTS OF NUCLEAR EXPLOSIONS

VOLUME V: THEORETICAL DEVELOPMENTS

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SUMMARY

This fifth and final volume of the report, "Hydrodynamic Effects of Nuclear Explosions," presents new theoretical developments for two problems. The first is the determination of the waves resulting from the passage of a high-pressure disturbance over the free surface of a body of water. This would occur in the case of a burst on or over land near a shore and is, therefore, of interest to the Five City Study in which three nuclear surface bursts are near rivers or bays.

The second topic is motion of the ground water table induced by a surface burst. This problem is of interest for the determination of the migration of radioactive contaminants and is also applicable to three cities of the Five City Study.

Only theoretical development is given here. Typical methods of application may be found in Volume IV of this report subtitled "Five City Study."

CONTENTS

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NOMENCLATURE

Part I

a	=	decay constant in error function
f	=	Laplace transform function
f_n	=	friction factors
g	=	acceleration of gravity
h	=	water depth
k	=	variable in Hankel transform
m, n	=	summation indices
s	=	variable in Laplace transform
t	=	time
u	=	horizontal particle velocity
x	=	horizontal distance from shore
erf	=	error function
erfc	=	complimentary error function
A	=	arbitrary constant
F	=	Laplace-Hankel transform of dimensionless surface elevation
$F(a, b, c, z)$	=	hypergeometric function
G	=	function
H	=	Heaviside step function
I	=	integral
J_n	=	Bessel function of nth order
L	=	length scale
P	=	pressure acting upon surface
P_o	=	Magnitude of pressure step
P_n	=	Legendre function
P^*	=	pressure front
$\mathbb{P}$	=	Laplace transform of dimensionless pressure
Q	=	Laplace-Hankel transform dimensionless pressure
Q	=	magnitude of particle velocity vector

NOMENCLATURE
(Continued)

S = variable
T = time scale
U = horizontal velocity of pressure front
U_A = wind velocity
α = beach slope
β = exponential decay factor
γ = dimensionless pressure
δ = Dirac delta function
ζ = dimensionless surface elevation
ζ_d = dynamic response
η = surface elevation
θ = τ
ρ = fluid density
ρ_A = air density
τ = dimensionless time
τ_s = shear stress at surface
τ_b = shear stress at bottom
χ = transcendental function
Γ = contour of integration
Γ = gamma function
Δ = variable in transform
$()_t$, $()_x$ etc. refer to partial derivatives
$()_D$ refers to peak depression
$()_E$ refers to peak elevation

Part II

d = distance between well and reference point
f, g = various functions as required in analyses
h = water depth

NOMENCLATURE
(Concluded)

ζ	=	dimensionless water level ratio
θ	=	angle
λ	=	distance between crater and river
ν	=	variable
ρ	=	fluid density
ρ	=	H/R
φ	=	potential function
∇	=	gradient operator
∇^2	=	Laplacian operator
$()^*$		refers to dimensionless variables
$()_o$		refers to initial value
$()_w$		refers to transient water level
$()_s$		refers to stream water level
$()_n$		refers to successive terms in series solution
$()_t$, $()_r$, etc.		refers to partial derivatives

NOMENCLATURE
(Continued)

k = constant in Darcy law
m = porosity of soil
n = summation index
p = pressure
q = unit discharge
r, θ, z = cylindrical coordinates
s = drawdown of well
t = time
u = particle velocity component in r-direction
w = particle velocity component in z-direction
w, x, y, z = dummy variables as required
A_p = particle radius of bed material
A_n, B_n = constants
C = concentration
C, D = constants
D = Domain
Dx, Dy = dispersion coefficients
E, F = variable functions
H = height of fluid at $r = \infty$
I_o, K_o = Bessel functions of 1st and 2nd kinds, respectively
K = permeability
Q = discharge rate
R = cavity, crater or well radius
S_f = free surface
T = hydraulic transmissivity
U = velocity
α, β = variables
γ = specific weight of fluid
∇ = datum
ϵ = specific yield = effective porosity

PART I

WAVES GENERATED BY A TRAVELING DISTURBANCE

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1. ESTABLISHMENT OF THE BASIC EQUATIONS

Consider a two-dimensional pressure disturbance arriving at the free surface of a body of water, as illustrated in Fig. 1. The response of the water to such a disturbance will be analyzed by means of the long wave equations valid for shallow water:

Continuity:

$$\eta_t + \left[u (h + \eta) \right]_x = 0 \tag{1}$$

Momentum:

$$u_t + u u_x = -g \eta_x - \frac{1}{\rho} P_x - \frac{\tau_s - \tau_b}{\rho (h + \eta)} \tag{2}$$

where

- x = distance from shore
- t = time after arrival of pressure at shore
- h = $h(x)$ = still water depth
- η = surface elevation around still water level
- u = horizontal component of water particle velocity, assumed constant over a vertical
- g = acceleration due to gravity
- ρ = water density
- P = pressure acting on the surface
- τ_s = shearing stress at the surface
- τ_b = shearing stress at the bottom

and subscripts are used to denote partial differentiation with respect to themselves.

Of couse, it is possible to treat these equations successfully by numerical techniques (finite differences, the method of characteristics). However, through suitable approximation, it is possible to derive analytic solutions, in certain instances, which are sufficient.

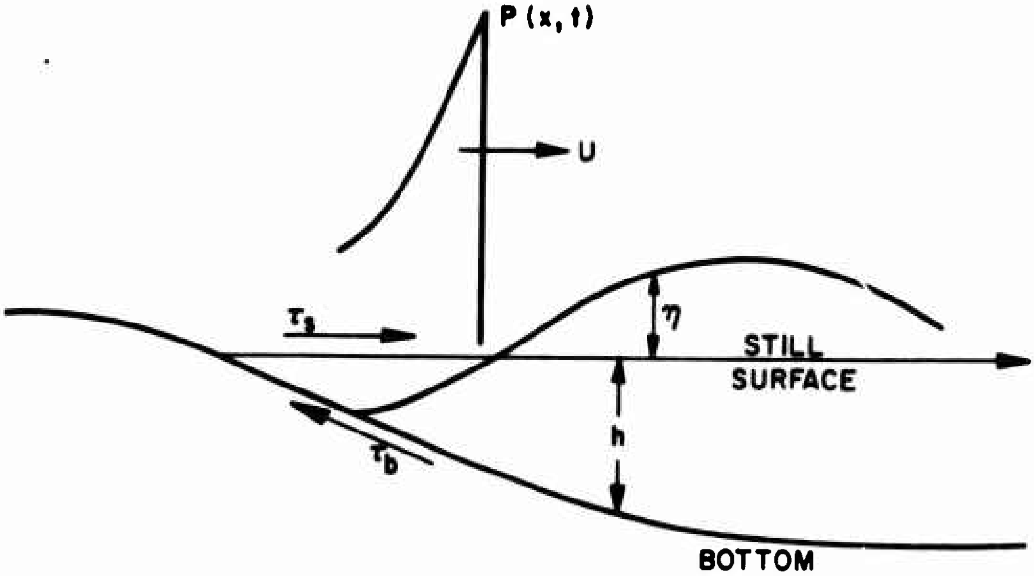

Figure 1
Problem configuration

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The shearing stresses, τ_s and τ_b, may be written

$$\tau_s = \rho_A f_1 U_A^2$$

$$\tau_b = \rho f_2 u^2$$

where ρ_A is the air density and U_A is the wind velocity near the surface. Although U_A may be large, ρ_A and f_1 are small so that τ_s may be neglected. Similarly, although ρ is large, f_2 and u are small so that τ_b may be neglected. Hence, the shearing stress term will be neglected; a good approximation except near the shore where $h + \eta = 0$.

The equations may be linearized by assuming the convective inertia term, uu_x, to be small and that η may be neglected in comparison with h (a crude approximation near the shore). Then the equations become:

Continuity:

$$\eta_t + (uh)_x = 0 \tag{3}$$

Momentum:

$$u_t = - g\eta_x - \frac{1}{\rho} P_x \tag{4}$$

Differentiating Eq. 3 by $\partial/\partial t$ and Eq. 4 by $\partial/\partial x$ gives

$$\eta_{tt} + u_t h_x + u_{xt} h = 0 \tag{5}$$

and

$$u_{xt} = - g\eta_{xx} - \frac{1}{\rho} P_{xx} \tag{6}$$

Eliminating u_t and u_{xt} from Eq. 5 by means of Eq. 4 and Eq. 6 results in an equation for η:

$$\eta_{tt} - h_x \left(g\eta_x + \frac{1}{\rho} P_x\right) - h \left(g\eta_{xx} + \frac{1}{\rho} P_{xx}\right) = 0 \qquad (7)$$

For a uniformly sloping beach, $h = \alpha x$, one has

$$x\,\eta_{xx} + \eta_x - \frac{1}{\alpha g}\,\eta_{tt} = -\frac{1}{\rho g}\,(xP_{xx} + P_x) \qquad (8)$$

By defining time and length scales

$$T \equiv U/\alpha g \qquad (9)$$

$$L \equiv U^2/\alpha g \qquad (10)$$

where U is some (constant) typical velocity, one may introduce the dimensionless variables

$$\left.\begin{aligned} \zeta &\equiv \eta/L \\ \chi &\equiv x/L \\ \tau &\equiv t/T \\ \gamma &\equiv P/\rho g L \end{aligned}\right\} \qquad (11)$$

which, when substituted in Eq. 8, gives

$$\chi\,\zeta_{\chi\chi} + \zeta_\chi - \zeta_{\tau\tau} = -(\chi\,\gamma_{\chi\chi} + \gamma_\chi) \qquad (12)$$

Assuming no initial disturbance, the initial conditions are

$$\zeta(\chi, 0) = \zeta_\tau(\chi, 0) = 0 \qquad (13)$$

The boundary condition is imposed that

$$\zeta \to 0 \qquad \text{as} \qquad \chi \to \infty \tag{14}$$

The general solution of this initial boundary value problem can be found by first applying the Laplace transform to τ, making the change of variable

$$\xi = \sqrt{\chi}$$

and then applying the Hankel transform of zeroth order to ξ.

The Laplace transform of Eq. 12 with respect to τ is

$$\chi f'' + f' - s^2 f = -(\chi \mathbb{P}'' + \mathbb{P}') \tag{15}$$

where

$$f(\chi, s) = \int_0^\infty \zeta(\chi, \tau)\, e^{-s\tau}\, d\tau \tag{16}$$

$$\mathbb{P}(\chi, s) = \int_0^\infty \gamma(\chi, \tau)\, e^{-s\tau}\, d\tau \tag{17}$$

and primes denote differentiation with respect to χ.

Introducing the change of variable

$$\xi = \sqrt{\chi} \tag{18}$$

Eq. 15 becomes

$$\frac{d^2 f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} - 4s^2 f = -\left(\frac{d^2\mathbb{P}}{d\xi^2} + \frac{1}{\xi}\frac{d\mathbb{P}}{d\xi}\right) \tag{19}$$

Letting

$$F(k) = \int_0^\infty f(\xi)\, \xi\, J_o(k\xi)\, d\xi \tag{20}$$

$$Q(k) = \int_0^\infty P(\xi)\, \xi\, J_o(k\xi)\, d\xi \tag{21}$$

and using the property

$$\int_0^\infty \left(\frac{d^2 f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi}\right) \xi\, J_o(k\xi)\, d\xi = - k^2 F \tag{22}$$

the Hankel transform of Eq. 19 gives

$$F = - \frac{k^2 Q}{k^2 + 4s^2} \tag{23}$$

Then the solution obtained by inversion transforms is found to be

$$\zeta(\chi, \tau) = - \gamma(\chi, \tau) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, e^{s\tau} \int_0^\infty dk\, k\, J_o(k\xi) \frac{4s^2 Q}{(k^2 + 4s^2)} \tag{24a}$$

The first part of this expression clearly represents the hydrostatic response, whereas the integral term represents the dynamic response which will be denoted by ζ_d. That is,

$$\zeta(\chi, \tau) = - \gamma(\chi, \tau) + \zeta_d \tag{24b}$$

Before presenting detailed analyses of this solution in specific cases, it is of interest to present a simplified calculation of the small time response. Neglecting the gravity terms in Eq. 7 results in

$$\zeta_{\tau\tau} \approx \chi\, \gamma_{\chi\chi} + \gamma_\chi \tag{25}$$

A pressure wave possessing a sharp front moving at speed U in the positive χ direction can be represented as:

$$\gamma(\chi, \tau) = \gamma(\tau - \chi) = H(\tau - \chi)\, P^*(\tau - \chi) \tag{26}$$

where H is the Heaviside step function and P^* is continuous. Because of the form of P^*, one may replace $\partial\gamma/\partial\chi$ by $-(\partial\gamma/\partial\tau)$ in Eq. 25:

$$\zeta_{\tau\tau} \approx \chi\, \gamma_{\tau\tau} - \gamma_{\tau} \tag{27}$$

Making use of Eq. 13 and the identity

$$\int_0^{\tau} H(\tau - \chi)\, P^*(\tau - \chi)\, d\tau = H(\tau - \chi) \int_0^{\tau - \chi} P^*(\xi)\, d\xi \tag{28}$$

one obtains

$$\zeta \approx H(\tau - \chi) \left\{ \chi\, P^*(\tau - \chi) - \int_0^{\tau - \chi} P^*(\xi)\, d\xi \right\} \quad \tau << 1 \tag{29}$$

This may be generalized for an arbitrary bottom $y = -h(\chi)$ and any pressure distribution traveling unchanged at constant speed, $\gamma = \gamma(\tau - \chi)$:

$$\zeta \approx h(\chi)\, \gamma(\tau - \chi) - h_{\chi}(\chi) \int_0^{\tau} \gamma(\tau - \chi)\, d\tau \quad \tau << 1 \tag{30}$$

If γ has a sharp front ahead of which the pressure is zero

$$\zeta \approx H(\tau - \chi) \left\{ h(\chi)\, P^*(\tau - \chi) - h_{\chi}(\chi) \int_0^{\tau - \chi} P^*(\xi)\, d\xi \right\} \quad \tau << 1 \tag{31}$$

These expressions are a first approximation only but are useful since P^* and h may be quite arbitrary. Improvements of a higher order can be obtained by investigating the full boundary value problem.

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2. STEP-FUNCTION PRESSURE OVER A UNIFORMLY SLOPING BEACH

The simplest model of the pressure assumes a sharp front moving at unit velocity, the pressure being zero ahead and unity behind. In other words

$$\gamma(\chi, \tau) = H(\tau - \chi) = H(\tau - \xi^2) \tag{32}$$

where H is the Heaviside step function. The sharp front of this model is a realistic representation of the traveling shock, although the pressure behind the shock will actually decay. Such a decaying pressure will be treated in Section 3.

The composite Laplace-Hankel transform of γ is

$$Q(k, s) = \frac{1}{2s^2} \exp\left(- \frac{k^2}{4s}\right) \tag{33}$$

Inserting this expression into Eq. 24 it is found that

$$\zeta_d = 2 \int_0^\infty k\, J_o(k\xi) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp\left(s\tau - \frac{k^2}{4s}\right)}{k^2 + 4s^2}\, ds\, dk \tag{34}$$

We may evaluate the Laplace inversion integral first and then deal with the Hankel integral. Although the behavior of the Laplace integral for large τ can be found by the method of steepest descent, as will be discussed in Section 5, the results are not too significant, physically. Here we shall be concerned with the small time response only.

The response at small τ is represented by the transform at large s. Therefore, we expand the integrand for large s:

$$I \approx \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n \geq 1} \frac{(-k^2)^{n-1}}{(4s^2)^n} \exp\left(s\tau - \frac{k^2}{4s}\right) ds \tag{35}$$

Each integral in the expansion may be evaluated:

$$I \approx \frac{1}{2} \sum_{n \geq 1} (-1)^{n-1} \frac{\tau^{n-1/2}}{k} J_{2n-1} (k \sqrt{\tau}) \tag{36}$$

so that:

$$\zeta_d \approx \sum_{n \geq 1} (-1)^{n-1} \tau^{n-1/2} \int_0^\infty J_o(k\xi) J_{2n-1} (k \sqrt{\tau}) \, dk \tag{37}$$

The integral factor here may be evaluated in terms of Gauss' hypergeometric function:

$$\int_0^\infty J_o(k\xi) J_{2n-1} (k\sqrt{\tau}) \, dk = \tau^{-1/2} F\left(n, -n+1, 1, \frac{\chi}{\tau}\right) \qquad \chi < \tau$$

$$= 0 \qquad \chi > \tau$$

which, in turn, may be expressed in terms of Legendre polynomials:

$$\int_0^\infty J_o(k\xi) J_{2n-1} (k\sqrt{\tau}) \, dk = H(\tau - \chi) \, \tau^{-1/2} P_{n-1}\left(1 - 2\frac{\chi}{\tau}\right) \tag{38}$$

Inserting this result into Eq. 37 gives

$$\zeta_d \approx \sum_{n \geq 0} (-\tau)^n P_n\left(1 - 2\frac{\chi}{\tau}\right) \qquad \chi < \tau$$

$$= 0 \qquad \chi > \tau \tag{39}$$

Combining this with the static component $-\gamma = -H(\tau - \chi)$ gives

$$\zeta \approx \left\{ \sum_{n \geq 1} (-\tau)^n P_n\left(1 - 2\frac{\chi}{\tau}\right)\right\} H(\tau - \chi) \qquad \tau << 1 \tag{40}$$

This series may be summed explicitly, giving:

$$\left.\begin{aligned} \zeta &\approx (1 + 2\tau + \tau^2 - 4\chi)^{-1/2} - 1 & \chi < \tau \\ &= 0 & \chi > \tau \end{aligned}\right. \tag{41}$$

Since τ and χ are very small compared to unity, this expression may be approximated as

$$\left.\begin{aligned} \zeta &\approx 2\chi - \tau & \chi < \tau \\ &= 0 & \chi > \tau \end{aligned}\right\} \tag{42}$$

which is seen to be the first term of the series in Eq. 40, the succeeding terms being small. As illustrated in Fig. 2, the free surface is deformed only beneath the pressure, having a sharp front at the pressure front. The mathematical profile OA'CD satisfies conservation of mass. Due to the presence of the bottom, however, the physical system cannot; perhaps the new shoreline at A should be interpreted as a sink-like singular point. Singular behavior at a shoreline has been investigated by Ho and Meyer (1962).

Reverting to physical variables, it is found that the peak elevation, η_E, and the peak depression, η_D, are

$$\left.\begin{aligned} \eta_E &\approx \frac{P_o \alpha t}{\rho U} \\ \eta_D &\approx -\frac{P_o \alpha^2 t}{\rho U(2 + \alpha)} \end{aligned}\right\} \tag{43}$$

where P_o is the magnitude of the pressure step.

This case may be extended to consider a step of finite duration. The pressure may be thought of as a superposition of two step functions:

$$\gamma(\chi, \tau) = H(\tau - \chi) - H(\tau - \tau_o - \chi) \tag{44}$$

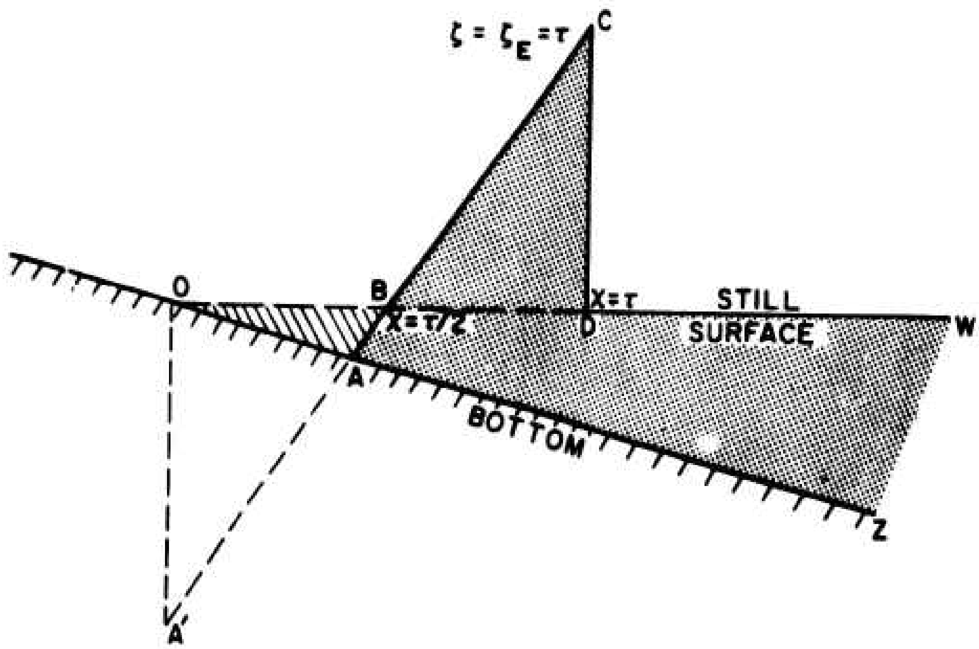

Figure 2
Step-pressure response

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Analysis similar to the foregoing yields the result illustrated in Fig. 3; the length AE depends on τ and τ_o and is an almost uniform depression. Again, the maximum water elevation is $\zeta_E = \tau$.

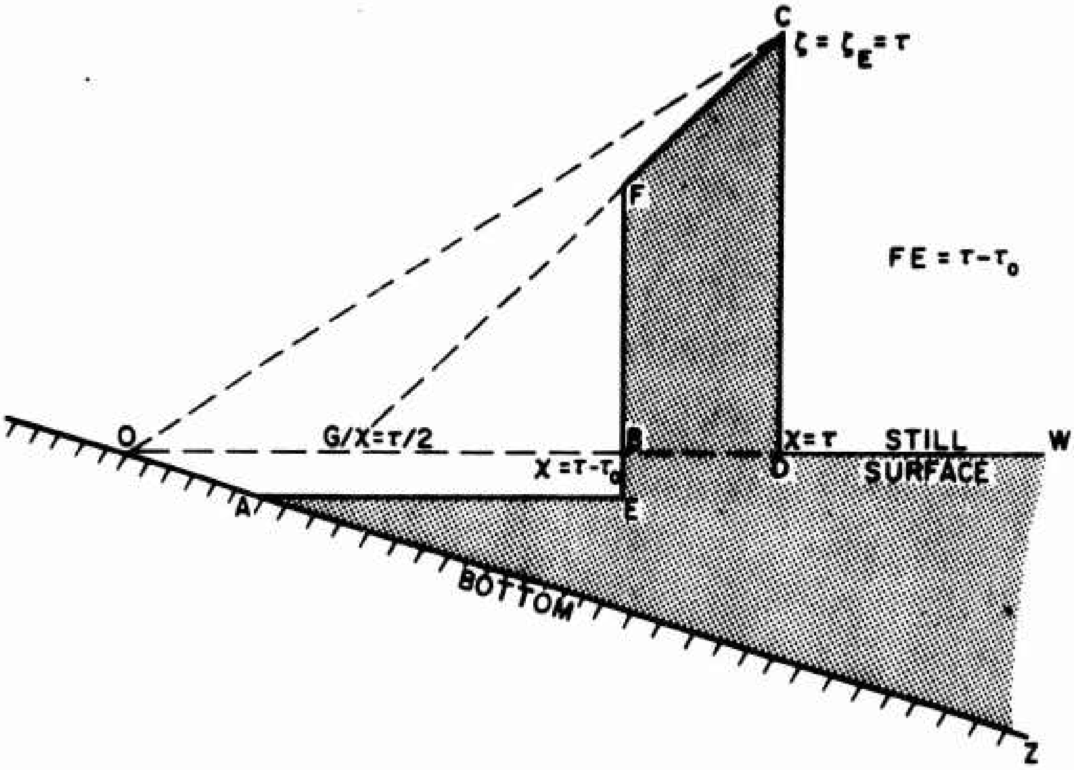

Figure 3
Finite duration step-pressure response

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3. DECAYING PRESSURE OVER A UNIFORMLY SLOPING BEACH

In this second model, we assume a pressure distribution that decays behind a sharp front, as:

$$\gamma(\chi,\tau) = H(\tau - \chi)\, e^{-\beta(\tau-\chi)} \tag{45}$$

Then we have

$$Q(k,s) = \frac{\exp\left(-\frac{k^2}{4s}\right)}{2s(s+\beta)} = \frac{\exp\left(-\frac{k^2}{4s}\right)}{2s^2}\left(1 - \frac{\beta}{s+\beta}\right) \tag{46}$$

The first part of this expression is the same as that of Eq. 33 and so represents the dynamic response of that model, which we may denote by $\zeta_d^{(1)}$, enabling us to write

$$\zeta_d = \zeta_d^{(1)} - \beta \int_0^{\tau} \zeta_d^{(1)}(\chi,\tau^*)\, e^{-\beta(\tau-\tau^*)}\, d\tau^* \tag{47}$$

by the convolution theorem.

Again concentrating on behavior at initial moments $(\tau << 1)$ we may use the small-time approximation of $\zeta_d^{(1)}$, i.e., Eq. 39 to evaluate the convolution integral:

$$- \beta \int_0^{\tau} \zeta_d^{(1)}(\chi,\tau^*)\, e^{-\beta(\tau-\tau^*)}\, d\tau^*$$

$$= - \beta H(\tau - \chi) \int_{\chi}^{\tau} (1 + 2\tau^* + \tau^{*2} - 4\chi)^{-1/2}\, e^{-\beta(\tau-\tau^*)}\, d\tau^*$$

Expanding the radical and neglecting terms above first order (since $\tau << 1$) gives

$$\zeta_d = \zeta_d^{(1)} - H(\tau - \chi)\left\{\left(1 + \frac{1}{\beta}\right)\left(1 - e^{-\beta(\tau-\chi)}\right)\right.$$

$$\left. + 2\chi - \tau - \chi e^{-\beta(\tau-\chi)} + O(\tau^2)\right\} \tag{48}$$

Introducing $\zeta_d^{(1)}$ and $-\gamma$ then yields

$$\zeta = H(\tau - \chi)\left\{-\frac{1}{\beta}\left[1 - e^{-\beta(\tau-\chi)}\right] + \chi e^{-\beta(\tau-\chi)} + O(\tau^2)\right\} \tag{49}$$

This response is illustrated in Fig. 4 and may be somewhat more realistic than the previous models. Again, the peak height is $\zeta = \tau$.

Reverting to physical variables where P_o is the peak pressure and β is written as T/T_o, where T_o is the time scale of the pressure decay, gives

$$\eta_E \approx \frac{P_o \alpha t}{\rho U}$$
$$\eta_D \approx \frac{P_o \alpha T_o}{\rho U} \tag{50}$$

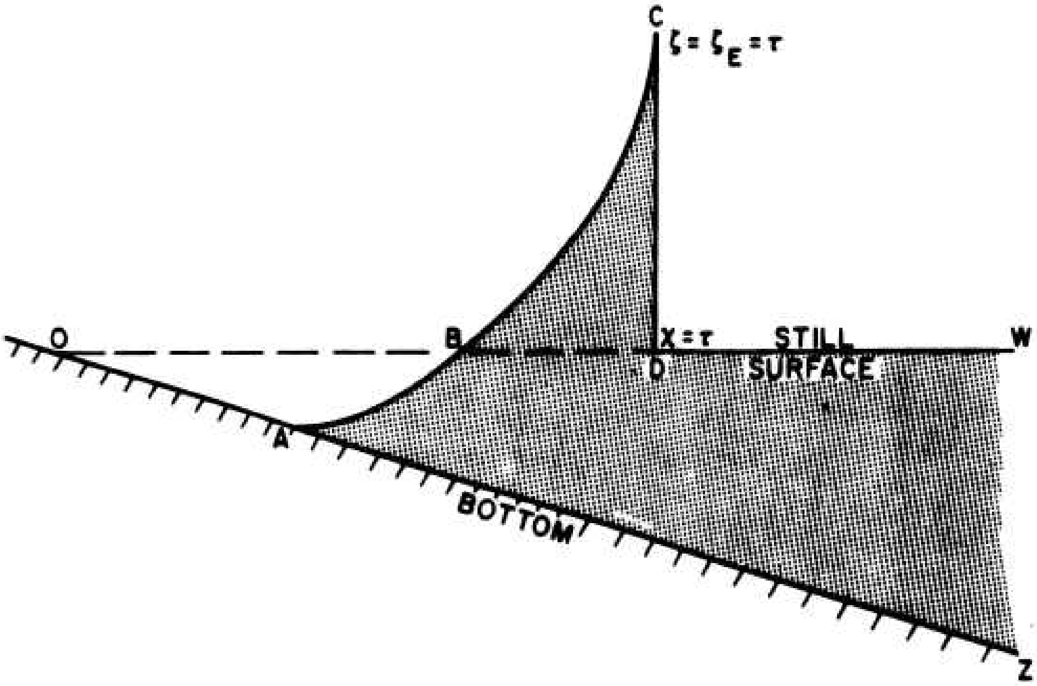

Figure 4

Decaying pressure response

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4. THE CASE OF A CONSTANT-DEPTH CHANNEL

The governing equation for the disturbance in a constant-depth channel with a wall at the leading end follows immediately from Eq. 7

$$h\,\eta_{xx} - \frac{1}{g}\eta_{tt} = -\frac{h}{\rho g} P_{xx} \qquad (51)$$

The boundary conditions may be written as

$$\begin{aligned} \eta_x &= 0 \quad \text{at} \quad x = 0 \quad (\therefore\ u = \frac{\partial\phi}{\partial x} = 0) \\ \eta &\to 0 \quad \text{as} \quad x \to \infty \quad \text{for} \quad t < \infty \end{aligned} \qquad (52)$$

while the initial conditions are

$$\eta = 0 = \eta_t \quad \text{at} \quad t = 0 \qquad (53)$$

For small time, an approximate result is again found by neglecting gravity forces so that Eq. 51 becomes

$$\eta_{tt} = \frac{h}{\rho} P_{xx} \qquad (54)$$

Writing the pressure as

$$P = P\left(t - \frac{x}{U}\right)$$

gives

$$P_{xx} = \frac{1}{U^2} P_{tt}$$

Letting P be a step function

$$P = P_o H\left(t - \frac{x}{U}\right) \qquad (55)$$

where P_o is the magnitude of the pressure step, we have

$$P_{xx} = \frac{P_o}{U^2} \delta' \left(t - \frac{x}{U}\right) \tag{56}$$

where δ is the delta function. Then it follows that

$$\eta_{tt} = \frac{P_o h}{\rho U^2} \delta' \left(t - \frac{x}{U}\right) \tag{57}$$

Integrating with respect to t from 0 to t and utilizing the initial conditions gives

$$\eta(x, t) = \frac{P_o h}{\rho U^2} \left\{ H\left(t - \frac{x}{U}\right) - H\left(- \frac{x}{U}\right) - t\, \delta\left(\frac{x}{U}\right) \right\} \tag{58}$$

On the other hand, if we take P to be a smooth function, we have

$$\eta_{tt} = \frac{h}{\rho U^2} P_{tt} \tag{59}$$

so that

$$\eta(x, t) = \frac{h}{\rho U^2} \left\{ P(x, t) - P(x, 0) - t\, P_t(x, 0) \right\} \tag{60}$$

For example, letting P be given by

$$P = P_o \operatorname{erfc} \left\{ a \left(\frac{x}{U} - t\right) \right\} \tag{61}$$

where erfc denotes the complementary error function, yields

$$\eta(x, t) = \frac{P_o h}{\rho U^2} \left\{ \operatorname{erfc} \left[a \left(\frac{x}{U} - t\right) \right] - \operatorname{erfc} \left(a \frac{x}{U} \right) - \frac{2a\, t}{\sqrt{\pi}} \exp \left[a^2 \left(\frac{x}{U}\right)^2 \right] \right\} \tag{62}$$

Figure 5 illustrates such a response.

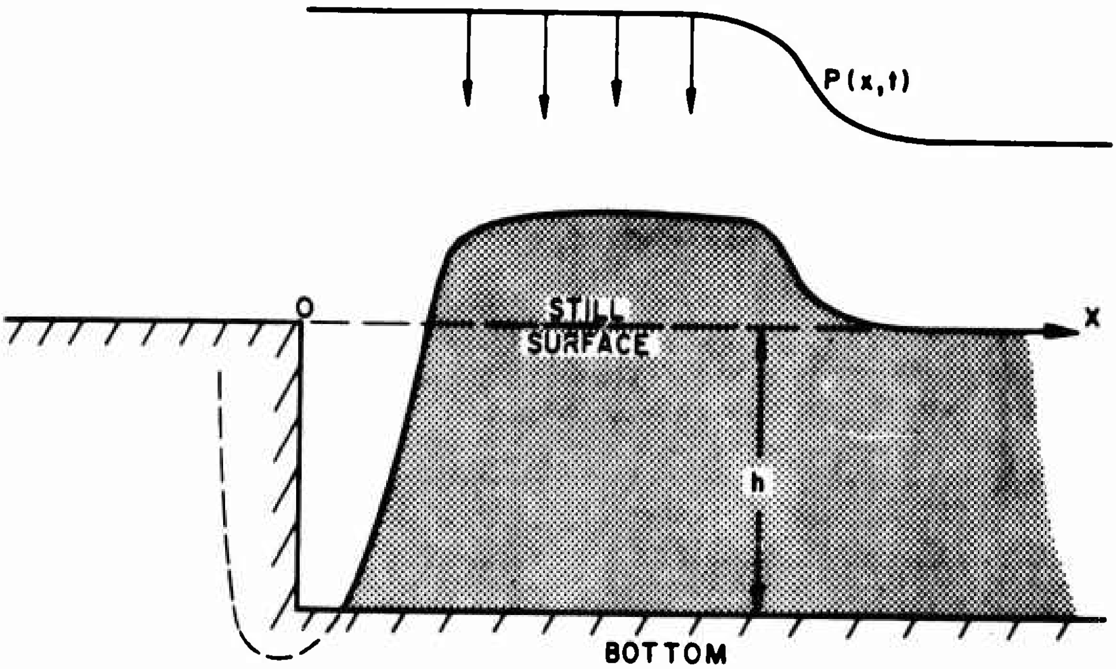

Figure 5

Response of constant-depth channel

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5. RESPONSE AT LARGE TIME

It is evident that the present model can only poorly describe the large time response since at large time the pressure front will be acting in water which is no longer shallow. Furthermore, as the blast wave progresses, its strength and velocity eventually decrease to those of an acoustic wave, so that our assumptions are invalid. But, still, it is of interest to see what the results are.

Denoting the Laplace inversion integral of Eq. 34 by I and making the change of variable

$$\Delta = S\theta \qquad \theta = \sqrt{\tau} \tag{63}$$

we have

$$I = \frac{1}{2\pi i}\left(\frac{\theta}{4}\right)\int_{c-i\infty}^{c+i\infty} \frac{e^{\theta(\Delta - k^2/4\Delta)}}{\Delta^2 + k^2\theta^2/4}\, d\Delta \tag{64}$$

This is of the form

$$\int_{\Gamma} e^{\theta f(\Delta)}\, G(\Delta, \theta)\, d\Delta$$

for which the method of steepest descent is suitable. When $\theta \gg 1$ ($\tau \gg 1$) one may evaluate the integral approximately by deforming the contour in the Δ plane to pass the saddle-points located at

$$\Delta = \frac{ik}{2},\ -\frac{ik}{2}$$

The steepest path should be directed at angles $3\pi/4$ and $\pi/4$ with the positive real axis in the complex Δ plane. Since $G = (\Delta^2 + k^2\theta^2/4)^{-1}$

has two simple poles at $\Delta = \pm ik\theta/2$ it is evident that the residues must be accounted for in changing the contour from Γ to Γ' (Fig. 6). Following standard formulas, we have:

$$I = \frac{1}{2\pi i}\int_{\Gamma} = \frac{1}{2\pi i}\int_{\Gamma'} + \text{Residues}$$

$$= I_{\text{saddle points}} + I_{\text{poles}}$$

$$\approx \frac{1}{2}\sqrt{\frac{\theta}{\pi}}\ \frac{k^{-3/2}}{\tau - 1}\ (\cos k\theta - \sin k\theta)$$

$$+ \frac{1}{2k}\sin\left(\frac{\tau + 1}{2}\ k\right) \quad \tau >> 1 \tag{65}$$

Equation 65 may be substituted in Eq. 34 to yield

$$\zeta_d \approx \frac{\tau^{1/4}}{\tau - 1}\ \frac{1}{\sqrt{\pi}}\int_0^{\infty}(\cos k\theta - \sin k\theta)\, J_o(k\xi)\frac{dk}{\sqrt{k}}$$

$$+ \int_0^{\infty}\sin\left(k\frac{\tau + 1}{2}\right) J_o(k\xi)\, dk \tag{66}$$

Both integrals in this expression may be explicitly evaluated:

$$(\zeta_d)_{\text{poles}} = \int_0^{\infty}\sin\left(k\frac{\tau + 1}{2}\right) J_o(k\xi)\, dk$$

$$= \left[\left(\frac{\tau + 1}{2}\right)^2 - x\right]^{-1/2} \quad \chi < \left(\frac{\tau + 1}{2}\right)^2$$

$$= 0 \qquad \chi > \left(\frac{\tau + 1}{2}\right)^2 \tag{67}$$

3.2 Sample Analysis

The value of a sample is only as good as the water in it can be measured. This measurement is a function of how the water is removed from the sample chamber walls, the instrument used for analysis, and how accurately the instrument is calibrated. Although present procedures may be adequate and possibly even optimum, detailed quantitative investigations of many aspects of the procedures would provide a sounder basis for analysis results and may well show how improvements can be made.

3.2.1 Water Removal

Experiments on the sampler so far have shown that water "sticks" to the sample chamber walls and that heating drives the water off. The more water that sticks to the walls, the more margin for error exists in the analysis results since even small percentage differences in the amount that sticks in the preflight evacuation and the amount that sticks in the postflight analysis might mean large amounts of water. A program to quantitatively determine the effects of the following variables on water retention is proposed.

a) Materials. Present materials plus any others that a literature survey indicates might be good.

b) Chamber Temperature

c) Chamber Pressure

d) Water Concentration

Suitable chambers of candidate materials would be tested on the presently used mass spectrometer (unless a better instrument is found, as discussed in the following section). Chamber surface area and volume would be comparable to that of an actual sample chamber. Tests would be run at at least three values of each of the other three variables so that

and

$$(\zeta_d)_{\text{saddle points}} = H(\chi - \tau) \frac{(\tau/\chi)^{1/4}}{\sqrt{\pi}(\tau - 1)}$$

$$\cdot \left\{ \frac{\Gamma(1/4)}{\sqrt{2}\,\Gamma(3/4)} F\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{\tau}{\chi}\right) - \frac{\sqrt{2}\,\Gamma(3/4)}{\Gamma\left(\frac{1}{4}\right)} \left(\frac{\tau}{\chi}\right)^{1/2} F\left(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}, \frac{\tau}{\chi}\right) \right\} \tag{68}$$

The complete expression for ζ then is

$$\zeta = -\gamma + (\zeta_d)_{\text{saddle points}} + (\zeta_d)_{\text{poles}} \tag{69}$$

Because of the discontinuities, it is convenient to indicate the effective region of each component of this solution in a $\chi - \tau$ diagram, shown in Fig. 7.

It is seen from Fig. 7 that two fronts exist, one moving with the pressure front as represented by the ray $\chi = \tau$, and the other moving with ever increasing speed ahead of the pressure front. The variable speed of the outer front is given by

$$\frac{d\chi}{d\tau} = \frac{\tau + 1}{2} = \sqrt{\chi} \tag{70a}$$

or in physical variables

$$\frac{dx}{dt} = \sqrt{\alpha g x} \tag{70b}$$

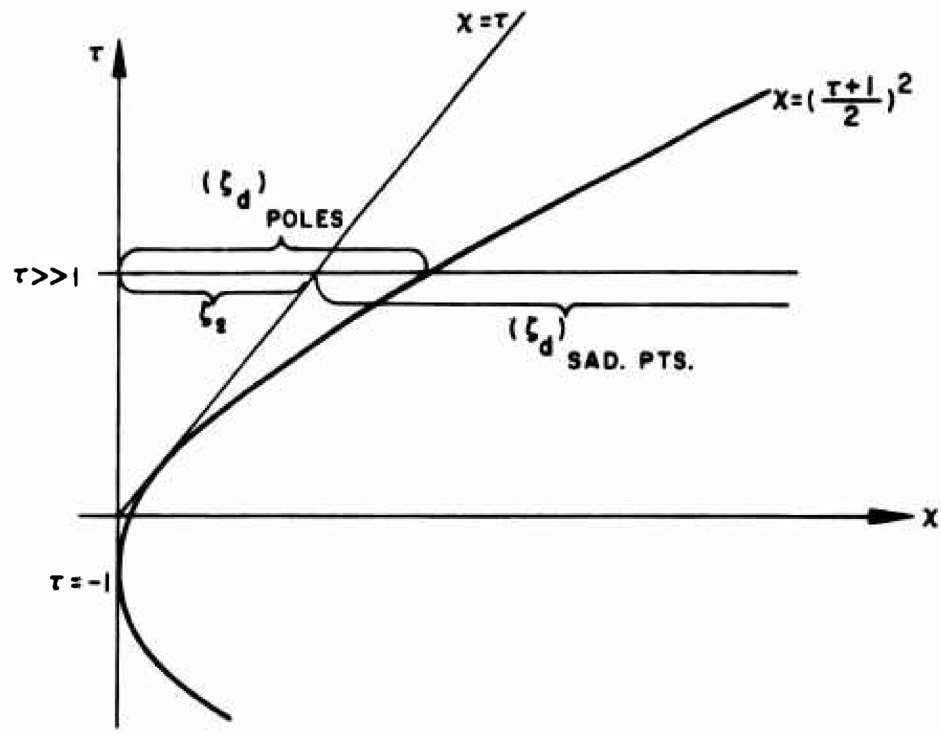

Figure 7

Regions of solution components

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which is precisely the local wave speed $\sqrt{gh}$ since the depth, h is αx. The wave height is infinite there (although integrably singular) and the surface in the neighborhood is essentially given by Eq. 67. Physically, the present model would predict a splash-like wave-front driven forward by the pressure front and traveling at the local wave speed.

The decreasing surface height as $\chi \to \infty$ is entirely due to the contribution of the saddle points, Eq. 68. By using the convergent expansion

$$F(a, b, c, z) = 1 + \sum_{n \geq 1} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{71}$$

Eq. 68 becomes

$$(\zeta_d)_{\text{saddle points}} \approx \frac{(\tau/\chi)^{1/4}}{\sqrt{\chi}\,(\tau - 1)} A\left\{1 - \frac{1}{A^2}\left(\frac{\tau}{\chi}\right)^{1/2} + O\left(\frac{\tau}{\chi}\right)\right\} \tag{72}$$

where

$$A = \frac{\Gamma(1/4)}{\sqrt{2}\,\Gamma(3/4)} \tag{73}$$

and τ/χ is small. Near the pressure front, $\tau/\chi \approx 1$, the following expansion may be used:

$$F(a, b, a + b, z) = - \frac{\Gamma(a + b)}{\Gamma(a)\,\Gamma(b)} \sum_{n \geq 0} \frac{(a)_n (b)_n}{(n!)^2}$$

$$\cdot\ (1 - z)^n \{\log(1 - z) - 2\psi(n + 1)$$

$$+ \psi(a + n) + \psi(b + n)\} \tag{74}$$

Then, recalling that $\psi(1 - m) = \psi(m) + \pi \cot \pi m$, it is found that at the front, $\tau / \chi = 1$,

$$(\zeta_d)_{\text{saddle points}} = \frac{1}{\tau - 1} \tag{75}$$

Combining these results, one arrives at the sketch shown in Fig. 8.

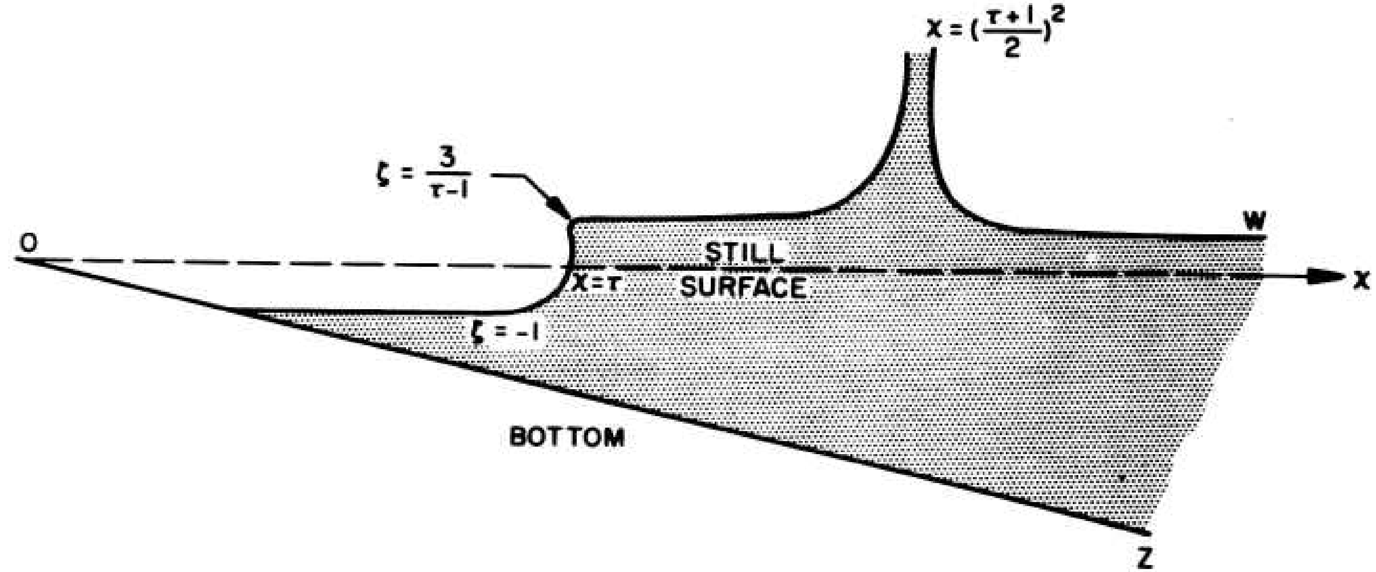

Figure 8
Large-time response

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6. DISCUSSION

In this analysis of a blast wave passing a shore two peculiarities arise: 1) the solution is not valid near the shoreline; 2) the free surface rises under a positive pressure. These are related and may be discussed in terms of a constant-depth channel which was treated in Section 4.

First, it is clear that the disturbance must occur wholly beneath the pressure distribution as long as the channel depth is small enough that the wave velocity is less than the pressure velocity. Lamb (1932) found that in the case of an infinitely long channel of depth h, a distributed pressure traveling at speed U will result in a surface change in phase with the pressure if $U < \sqrt{gh}$ and oppositive in phase if $U > \sqrt{gh}$. This may be seen as follows. Choosing a coordinate system moving with the pressure, one obtains a steady state. From the Bernoulli equation

$$\frac{P(x)}{\rho} + \frac{Q^2}{2\,gh^2(x)} + h(x) = \text{const}$$

it is found by differentiation that

$$\frac{1}{\rho}\,\frac{dP}{dx} + \left(1 - \frac{Q^2}{gh^3}\right)\frac{dh}{dx} = 0$$

$$\frac{dh}{dx} = \frac{\frac{1}{\rho}\,\frac{dP}{dx}}{\left(\frac{U^2}{gh} - 1\right)} \qquad \frac{Q}{h} = U$$

Hence, dh/dx and dP/dx are of $\left\{\begin{matrix}\text{same}\\ \text{opposite}\end{matrix}\right\}$ sign if $U \gtrless \sqrt{gh}$. It is then easy to see that a positive pressure gives rise to an elevation or a depression according to whether the flow is supercritical or subcritical. Since the flow velocity is just the velocity of the pressure in a stationary coordinate system, we have Lamb's result.

In the present case of a semi-infinite channel, part of the surface is elevated because $U > \sqrt{gh}$. However, due to conservation of mass, this fluid must be supplied from behind the pressure front. This is in accordance with the results presented earlier. The same considerations apply to the case of a sloping beach. The immediate neighborhood of the shore deserves further study.

PART II

GROUND WATER TABLE MOTION

1. INTRODUCTION

In the event of a nuclear explosion near the ground surface, a cavity would be produced. If the ground water level at the location of the nuclear explosion is relatively high, this would influence the flow of ground water. In particular, radioactive contamination of the ground water basin is possible. Previous investigation into deep-underground explosions indicates that water contamination is of little importance. The same, however, may not be true for a near-surface explosion.

Several problems are postulated within this report. These problems are either completely solved or solved to the point that numerical evaluation becomes straightforward. These problems are:

a) The filling of a cylindrical crater in an unconfined aquifer.

b) The filling of a cylindrical crater in an unconfined aquifer near a river.

c) The discharge of a stream into a large well in a confined aquifer.

d) A quiescent well in a uniform flow. The advection and diffusion of contaminants initially present in the well are also investigated.

e) Flow into a water surrounded crater.

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2. THE FILLING OF A CYLINDRICAL CRATER IN AN UNCONFINED AQUIFER

The problem under consideration is the flow of ground water into an empty cylindrical reservoir. The physical occurrence of this problem may be the filling of a cylindrical void excavated by a nuclear explosion. This situation is shown in Fig. 9.

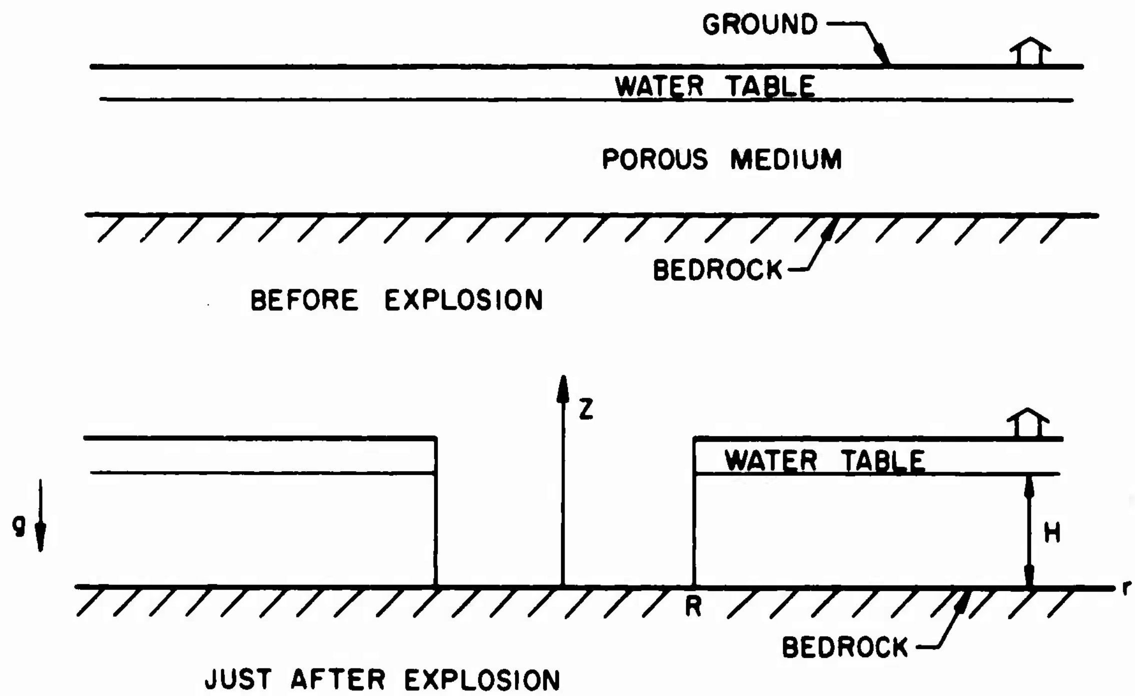

Figure 9

The material within a radius R of the explosion is assumed completely removed. The water in the ground starts to flow out of the wetted face of the walls of the crater and begins to fill the crater. As time approaches infinity, the crater would be filled with water to the level H. From the mathematical point of view, this problem is one of unsteady, unconfined flow radially towards a large "well" which is initially empty.

The problem is first formulated by writing down the differential equation and the boundary and initial conditions.

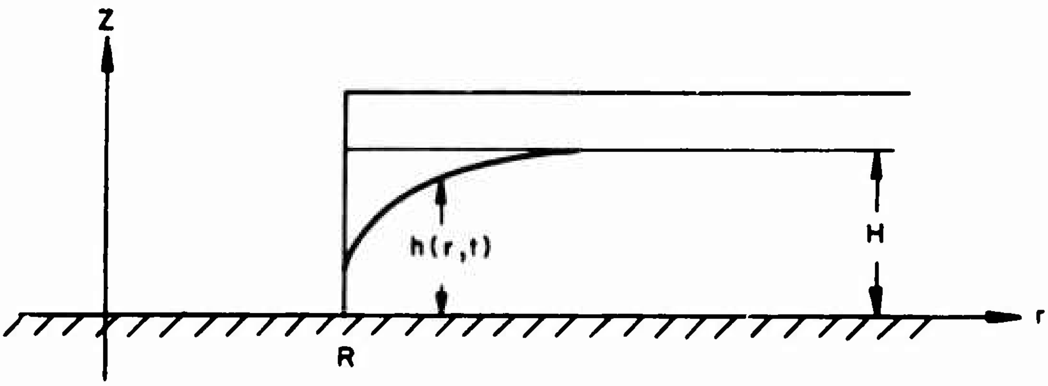

Figure 10

The differential equation is given by combining the equation of continuity

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

with Darcy's law

$$\vec{u} = \nabla \phi, \qquad \phi = -k\left[(p/\gamma) + z\right]$$

where axial symmetry $(\partial/\partial\theta = 0)$ has been assumed and

r = radial coordinate

z = vertical coordinate

u = velocity component in the r direction

w = velocity component in the z direction

$\vec{u}$ = velocity vector

φ = velocity potential

p = pressure

γ = specific weight

This gives

$$\nabla^2\varphi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2} = 0$$

This equation applies within the flow field which is bounded by the bottom bedrock, the top free surface, the left seepage face, and the height H at r = infinity as shown in Fig. 10.

The initial condition is the quiescent state with the free surface given by $z = H$ = constant for $R < r \leq \infty$.

The boundary condition at the right side $(r \to \infty)$ is the velocity $w \to 0$. The boundary condition at the bottom is $w = 0$ $(z = 0)$. The boundary condition at the seepage face $r = R$ $(0 < z < h_o)$ is $p = 0$ which implies $\varphi = -kz$.

The boundary condition at the free surface $z = h(r, t)$ is

a) kinematic

$$m\frac{\partial h}{\partial t} + \frac{\partial\varphi}{\partial r}\frac{\partial h}{\partial r} - \frac{\partial\varphi}{\partial z} = 0$$

b) dynamics

$$\varphi = -kz$$

where m is the porosity.

We now nondimensionalize the variables by the following formulas. The nondimensionalized variables are shown in Fig. 11.

$$h^* = h/H, \quad z^* = z/H, \quad r^* = r/R,$$

$$\varphi^* = \varphi/kH, \qquad t^* = \frac{t}{Hm/k}$$

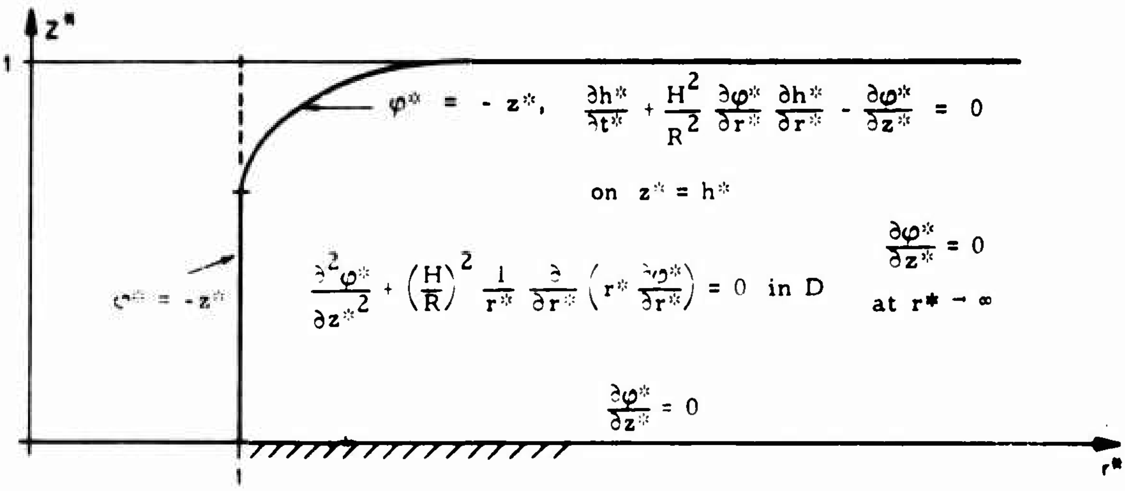

Figure 11

The two boundary conditions on the free surface may be combined. Ignoring the stars, we have

$$\varphi = -h \Rightarrow \varphi(r, z=h(r,t), t) = -h(r,t)$$

Differentiating

$$-\frac{\partial h}{\partial t} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial z}\frac{\partial h}{\partial t} \Rightarrow \frac{\partial \varphi}{\partial t} = -\frac{\partial h}{\partial t}\left(1 + \frac{\partial \varphi}{\partial z}\right)$$

$$-\frac{\partial h}{\partial r} = \frac{\partial \varphi}{\partial r} + \frac{\partial \varphi}{\partial z}\frac{\partial h}{\partial r} \Rightarrow \frac{\partial \varphi}{\partial r} = -\frac{\partial h}{\partial r}\left(1 + \frac{\partial \varphi}{\partial z}\right)$$

These may be used to eliminate one of the variables from the S_f (free surface) conditions.

2.1 Method I: Expansion-in-time Series

One of the standard ways of attacking this type of problem (initial boundary problem) is the expansion in powers of t. Thus, we let

$$\varphi(r, z, t) = \varphi_0(r, z) + t\varphi_1(r, z) + t^2\varphi_2 + \ldots$$

$$h(r, t) = h_0(r) + th_1(r) + \ldots$$

Substituting into the differential equation

$$\frac{\partial^2 \varphi_i}{\partial z^2} + \rho^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \varphi_i}{\partial r}\right) = 0 \qquad i = 0, 1, 2, \ldots$$

$$\rho = H/R = \text{constant}$$

The boundary conditions are

$$\frac{\partial \varphi_i}{\partial z} = 0 \qquad r \to \infty \qquad i = 0, 1, 2, \ldots$$

$$\frac{\partial \varphi_i}{\partial z} = 0 \qquad z = 0 \qquad i = 0, 1, 2, \ldots$$

$$\varphi_0 = -z \qquad \varphi_i = 0 \qquad i = 1, 2, \ldots \quad \text{on } r = 1, \; 0 < z < 1$$

$$\varphi_0 = -1, \quad \text{on } z = 1 \qquad 1 < r < \infty$$

For φ_i $i = 1, 2, \ldots$ on S_f, the condition will have to be derived by substituting the series into the boundary condition. The boundary condition is

$$- \frac{\dfrac{\partial \varphi}{\partial t}}{1 + \dfrac{\partial \varphi}{\partial z}} - \rho^2 \frac{\partial \varphi}{\partial r} \frac{\dfrac{\partial \varphi}{\partial r}}{1 + \dfrac{\partial \varphi}{\partial z}} - \frac{\partial \varphi}{\partial z} = 0$$

or

$$\frac{\partial \varphi}{\partial t} + \rho^2 \left(\frac{\partial \varphi}{\partial r}\right)^2 + \frac{\partial \varphi}{\partial z} + \left(\frac{\partial \varphi}{\partial z}\right)^2 = 0$$

Substituting the series into the boundary condition,

$$\varphi_1 + 2t\varphi_2 + \ldots + \rho^2 \left[\varphi_{0_r} + t\varphi_{1_r} + \ldots\right]^2 + \left[\varphi_{0_z} + t\varphi_{1_z} + \ldots\right]$$
$$+ \left[\varphi_{0_z} + t\varphi_{1_z} + \ldots\right]^2 = 0$$

Separating the various order of t

$$\varphi_1 = - \left\{\rho^2 \left(\frac{\partial \varphi_0}{\partial r}\right)^2 + \left(\frac{\partial \varphi_0}{\partial z}\right) + \left(\frac{\partial \varphi_0}{\partial z}\right)^2 \right\} \quad \text{on } z = 1$$

$$\varphi_2 = - \frac{1}{2} \left\{\rho^2 \left(2 \frac{\partial \varphi_0}{\partial r} \frac{\partial \varphi_1}{\partial r}\right) + \frac{\partial \varphi_1}{\partial z} + 2 \frac{\partial \varphi_0}{\partial z} \frac{\partial \varphi_1}{\partial z}\right\}$$

etc.

The advantage of this method is, as is the case with most approximate theories, that the nonlinear system is reduced to a succession of linear problems. The solutions should be valid for a relatively small t. Remembering that it is the dimensionless time which must be small, it is instructive to observe that

$$t^* = \frac{t}{Hm/k}$$

Typical values of H, m, and k indicate that, for $t^* = 0.1$, t is of the order of a day or so. Thus, the small time t^* solution is not entirely useless. Indeed, the solution $\varphi_0 + t\varphi_1$ should be good for a period of a week or so after initial blasting.

The problem for $\varphi_0(r, z)$ is, therefore, as follows (Fig. 12):

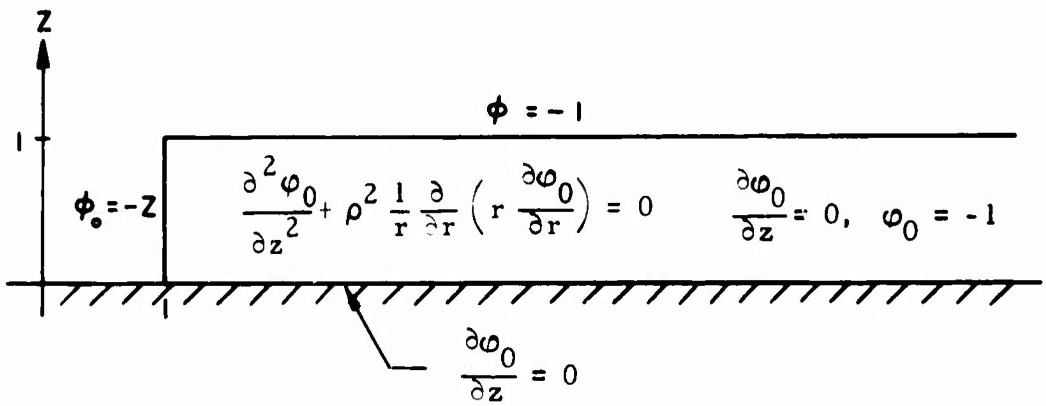

Figure 12

Before attempting a solution, let $\varphi_0 = -1 + \varphi_0'$ so that the problem in φ_0' is (Fig. 13)

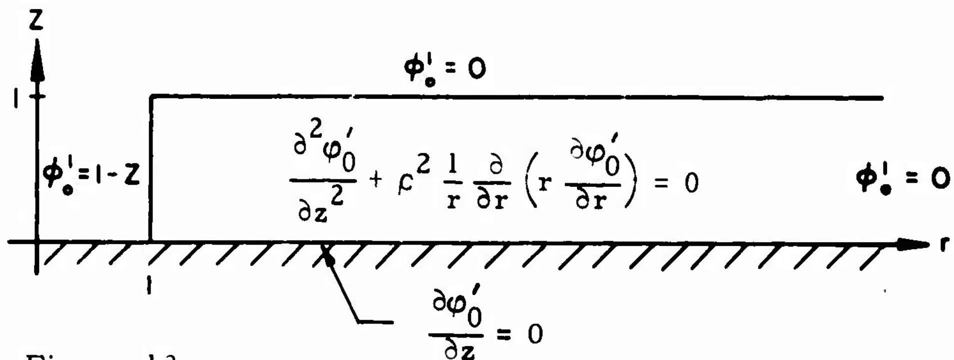

Figure 13

To obtain φ_0' we separate variables.

$$\varphi_0' = Z(z)\,R(r) \text{ to obtain}$$

$$\frac{Z''}{Z} = -\rho^2 \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \text{constant} = -k^2$$

$$Z'' + k^2 Z = 0 \quad \therefore \quad Z = \begin{Bmatrix} \sin kz \\ \cos kz \end{Bmatrix}$$

$$\frac{\partial \varphi_0'}{\partial z} = 0 \text{ at } z = 0 \therefore Z = A_k \cos kz$$

$$\varphi_0 = 0 \text{ at } z = 1 \therefore k = (n + \tfrac{1}{2})\pi, \quad n = 0, 1, 2, \ldots$$

The equation in R is

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dR}{dr}\right] - \frac{k^2}{\rho^2} R = 0$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - (k/\rho)^2 R = 0$$

which gives

$$R = C\, I_o\left(\frac{k}{\rho} r\right) + D\, K_o\left(\frac{k}{\rho} r\right)$$

where I_o and K_o are the modified Bessel functions of the 1st and 2nd kinds, respectively.

The boundary condition for $r \to \infty$ $(\varphi_0' \to 0)$ then requires $C = 0$ since

$$I_o(x) \sim \frac{1}{\sqrt{2\pi x}}\, e^{x}$$

and

$$K_o(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x}$$

Thus, solution may be expressed

$$\varphi_0 = -1 + \sum_{n=0}^{\infty} A_n\, K_o\left(\frac{(n+\frac{1}{2})\pi}{\rho} r\right) \cos\left(n + \frac{1}{2}\right)\pi z$$

It remains to evaluate A_n which may be obtained by the use of the remaining boundary condition, i.e.,

$$\text{at} \quad r = 1, \quad \varphi_0 = -z$$

$$\therefore \quad \sum_{n=0}^{\infty} A_n\, K_o\left(\frac{(n+\frac{1}{2})\pi}{\rho}\right) \cos\left(n + \frac{1}{2}\right)\pi z = 1 - z$$

Let

$$B_n = A_n\, K_o\left[\frac{(n+\frac{1}{2})\pi}{\rho}\right]$$

Then, by Fourier series methods, B_n is easily found to be

$$\frac{8}{\pi^2(2n+1)^2}.$$

Hence,

$$\varphi_0 = -1 + \sum_{n=0}^{\infty} \frac{8}{\pi^2(2n+1)^2} \frac{K_o\left(\frac{2n+1}{2}\frac{\pi r}{\rho}\right)}{K_o\left(\frac{2n+1}{2}\frac{\pi}{\rho}\right)} \cos(2n+1)\pi\frac{z}{2}$$

Thus, we now know $\varphi_0(r, z)$.

Let

$$f(r) = -\left\{\rho^2\left(\frac{\partial\varphi_0}{\partial r}\right)^2 + \left(\frac{\partial\varphi_0}{\partial z}\right)^2 + \frac{\partial\varphi_0}{\partial z}\right\}\Bigg|_{z=1}$$

The problem for φ_1 is then (Fig. 14)

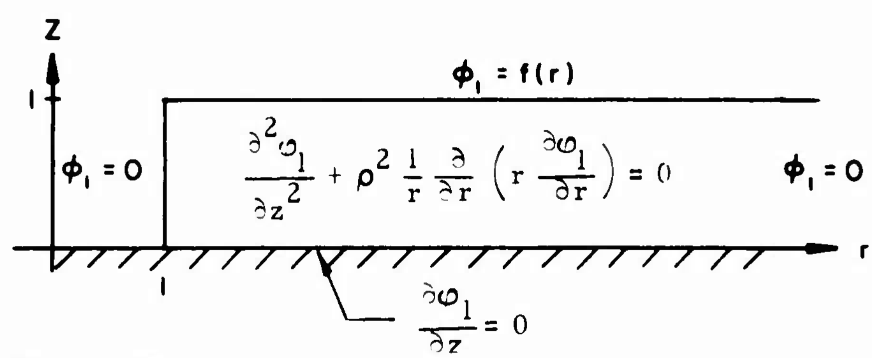

Figure 14

which may be solved by Hankel transform techniques. These will not be performed here.

2.2 Method II: Quasi-Steady Solution

The method presented before would yield the exact solution if the series were convergent and if the entire series were found. Practically, this should be convergent for a small enough t and, for such a small t, perhaps one or two terms in the series would be sufficient.
Even the first two terms, however, are not easily evaluated numerically, although it can certainly be done. This numerical difficulty would be more severe in the case of φ_1 than φ_0. In other words, even though the method of expansion in powers of t yields analytical solution in the form of $\varphi_0(r, z)$, $\varphi_1(r, z)$, etc., to obtain numerical results, it is still necessary to do a substantial amount of computational labor. Therefore, it would be desirable to have a simpler solution which would give numerical results without too much work. This would, of course, have to be a more approximate theory. Such a theory is presented. This theory is based on the assumption of quasi-steady motion and the Dupuit-Forscheimer approximation.

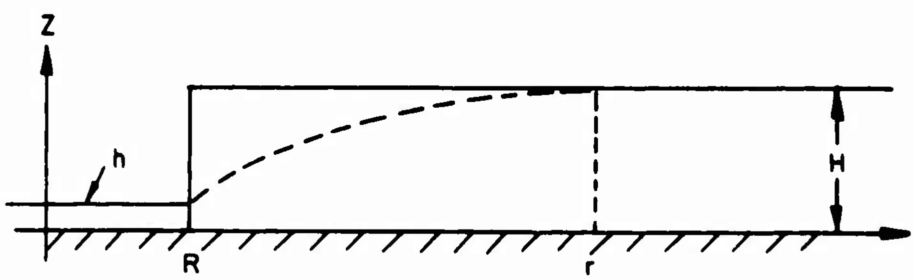

Figure 15

Let $Q(t)$ = the discharge through the sides

$h(t)$ = the height of water in the crater

H = the height of water in the porous medium for $r \to \infty$

At a given instant of time, it can be assumed that the motion is given by the Duprit-Forscheimer solution. With this assumption,

$$H^2 - h^2 = \frac{Q}{\pi K} \log r/R \tag{1}$$

where K is the permeability, and r is the radial distance representing the intersection of the Dupuit free surface with $z = H$.

There are now three unknowns $h(t)$, $Q(t)$ and $r(t)$. Therefore, two more equations are needed. One of them is obviously

$$\int_0^t Q(\tau)\, d\tau = \pi R^2 h$$

or

$$Q = \pi R^2 \frac{dh}{dt} \tag{2}$$

The second one is not so easily found. One way is that the volume of water in the reservoir should come out of the voids of the dewatered region. That is

$$\pi R^2 h = \epsilon \int_R^r 2\pi\rho\, s(\rho)\, d\rho \tag{3}$$

where s is the drawndown as illustrated in Fig. 8 and is given by the Dupuit formula

$$s(\rho) = H - \sqrt{H^2 - \frac{Q}{\pi K} \log \frac{\rho}{R}}$$

and ϵ is the porosity.

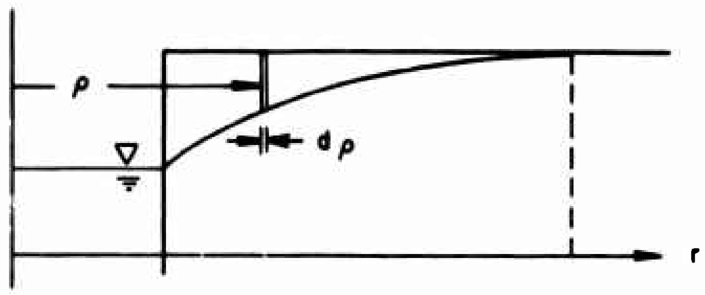

Figure 16

Before attempting a solution, nondimensionalize by the following variables

$$r^* = r/R, \quad h^* = h/H, \quad Q^* = Q/\pi K H^2, \quad t^* = \frac{t}{R^2/HK}$$

Then

$$1 - h^{*2} = Q^* \log r^*$$

$$Q^* = \frac{dh^*}{dt^*}$$

$$h^* = 2\epsilon \int_1^{r^*} \rho \left(1 - \sqrt{1 - Q^* \log \rho}\right) d\rho$$

$$= 2\epsilon \left[\frac{r^{*2}}{2} - \frac{1}{2}\right] - 2\epsilon \int_1^{r^*} \rho \sqrt{1 - Q^* \log \rho}\, d\rho$$

$$= \epsilon \left(r^{*2} - 1\right) - 2\epsilon \int_1^{r^*} \rho \sqrt{1 - Q^* \log \rho}\, d\rho$$

Dropping stars,

$$1 - h^2 = \frac{dh}{dt} \log r$$

$$h = \epsilon (r^2 - 1) - 2\epsilon \int_1^r \rho \sqrt{1 - \frac{dh}{dt} \log \rho} \; d\rho$$

This is a system of nonlinear integro-differential equations. First examine the integral

$$I = \int \rho \sqrt{1 - Q \log \rho} \; d\rho$$

Let

$$\rho = e^y \qquad d\rho = e^y \, dy$$

then

$$I = \int e^{2y} \sqrt{1 - Qy} \; dy$$

Now let

$$z = 1 - Qy, \quad y = (1 - z)/Q, \quad dy = - dz/Q$$

$$I = - \int \exp\left[\frac{2}{Q}(1-z)\right] \sqrt{z} \, \frac{dz}{Q} = \frac{-e^{2/Q}}{Q} \int \exp\left(-\frac{2}{Q} z\right) \sqrt{z} \; dz$$

Let

$$w = \frac{2}{Q} z, \qquad dz = \frac{Q}{2} dw, \quad w = x^2$$

$$I = - \frac{1}{Q} e^{2/Q} \left(\frac{Q}{2}\right)^{3/2} \int e^{-w} \sqrt{w} \; dw$$

$$= - \frac{e^{2/Q}}{Q} \left(\frac{Q}{2}\right)^{3/2} 2 \int e^{-x^2} x^2 dx$$

The integral $\int e^{-x^2} x^2 dx$ may be integrated by parts to give

$$\int e^{-x^2} x^2 dx = \int (x/2)\, e^{-x^2}\, 2x\, dx = \frac{x}{2} e^{-x^2} - \frac{1}{2}\int e^{-x^2}\, dx$$

The original integral I is evaluated between the limits r and 1. The relationship between x and ρ (after tracing through all the changes of variables) is

$$x = \sqrt{w} = \sqrt{2/Q}\sqrt{z} = \sqrt{2/Q}\;\sqrt{1 - Qy} = \sqrt{2/Q}\sqrt{1 - Q \log \rho}$$

Hence

$$I = \int_1^r \rho \sqrt{1 - Q \log \rho}\; d\rho$$

$$= \frac{x}{2} e^{-x^2} \Bigg|_{\sqrt{\frac{2}{Q}}\sqrt{1 - Q \log 1}}^{\sqrt{\frac{2}{Q}}\sqrt{1 - Q \log r}} - \frac{1}{2}\int_{\sqrt{2/Q}}^{\sqrt{2/Q}\sqrt{1 - Q \log r}} e^{-x^2}\, dx \, .$$

$$= \sqrt{\frac{1 - Q \log r}{2Q}}\, \exp\left[- \frac{2\,(1 - Q \log r)}{Q}\right] - \frac{1}{\sqrt{2Q}}\, e^{-2/Q}$$

$$- \frac{1}{2}\left\{\left(\sqrt{2/Q}\;\sqrt{1 - Q \log r}\right) F - \left(\sqrt{2/Q}\right) F\right\}$$

where $\qquad F(x) = \int_0^x e^{-\xi^2}\, d\xi$

To recapitulate, one has

$$1 - h^2 = Q \log r$$

$$Q = dh/dt$$

$$h = \epsilon(r^2 - 1) - \epsilon \Bigg\{ \sqrt{\frac{2 - 2Q \log r}{Q}} \exp\left[-\frac{(2 - 2Q \log r)}{Q} \right]$$

$$- \sqrt{2/Q}\; e^{-2/Q} \quad - \left[\left(\sqrt{\frac{2 - 2Q \log r}{Q}} \right) F \right.$$

$$\left. - \left(\sqrt{2/Q} \right) F \right] \Bigg\}$$

to be solved.

Now consider the function in the brackets for the expression in h.

$$E = \sqrt{\frac{2 - 2Q \log r}{Q}} \exp\left(- \frac{2 - 2Q \log r}{Q} \right) - \sqrt{2/Q}\; e^{-2/Q}$$

$$- \left(\sqrt{\frac{2 - 2Q \log r}{Q}} \right) F + \left(\sqrt{2/Q} \right) F$$

$$\sqrt{\frac{2 - 2Q \log r}{Q}} = \sqrt{2/Q}\, \sqrt{1 - Q \log r}$$

so that the expression E may be rewritten

$$\sqrt{2/Q}\, \sqrt{1 - Q \log r} \quad \exp\left[- \frac{2}{Q}(1 - Q \log r) \right] - \sqrt{2/Q}\; e^{-2/Q}$$

$$- \left(\sqrt{2/Q}\, \sqrt{1 - Q \log r} \right) F + \left(\sqrt{2/Q} \right) F$$

$$= \sqrt{2/Q}\; e^{-2/Q} \left\{ \sqrt{1 - Q \log r} \quad e^{2 \log r} - 1 \right\}$$

$$- \left(\sqrt{2/Q}\, \sqrt{1 - Q \log r} \right) F + \left(\sqrt{2/Q} \right) F$$

$$= \sqrt{2/Q}\ e^{-2/Q} \left\{ \sqrt{1 - Q \log r}\ r^2 - 1 \right\}$$

$$- (\sqrt{2/Q} \sqrt{1 - Q \log r}) F + (\sqrt{2/Q}) F$$

The system of nonlinear integro-differential equations can be solved, and the three functions $Q(t)$, $h(t)$ and $r(t)$ may be obtained. This involves some numerical computation; however, the amount of computation should be much less than the computation for φ_0 and φ_1, etc. of the first method.

Another method will be presented which is not based on Eq. 3 but is based on the formula

$$r = R + 1.5 \sqrt{\nu t} \tag{3a}$$

or

$$r^* = 1 + \frac{1.5\sqrt{\nu}}{R} \sqrt{\frac{R^2}{HK}} \sqrt{t^*} = 1 + 1.5\sqrt{\nu/HK}\sqrt{t^*}$$

where $\nu = K\bar{b}/\epsilon$. $\bar{b}$ is a weighted average of the depth of the flow which may, for all practical purposes, be taken as equal to H_0.

Letting $1.5\sqrt{\nu/HK} = \beta$, we have (dropping stars),

$$r = 1 + \beta\sqrt{t} \tag{3b}$$

Equation 3a simply states that the zone of influence of the draining crater spreads at a rate proportional to $\sqrt{t}$ (which is normal for diffusion-type phenomena) with a proportionality constant equal to $1.5\sqrt{\nu}$. This latter constant is based on observations of discharging wells, Hantush (1965).

The other two equations, of course, remain unchanged and are

$$1 - h^2 = Q \log r$$

$$Q = dh/dt$$

repeating Eq. 3b,

$$r = 1 + \beta\sqrt{t}$$

These combine easily to give

$$1 - h^2 = dh/dt \ \log (1 + \beta\sqrt{t})$$

which may be written

$$\frac{dh}{1 - h^2} = \frac{dt}{\log (1 + \beta\sqrt{t})}$$

The LHS is

$$\frac{1}{2}\left(\frac{1}{1 - h} + \frac{1}{1 + h}\right) dh$$

which integrates to

$$-\frac{1}{2} \log (1 - h) + \frac{1}{2} \log (1 + h) = \log \left(\frac{1 + h}{1 - h}\right)^{1/2}$$

The RHS is, letting $t = w^2$, $w' = \beta w$, $z = 1 + w'$ successively

$$\frac{2w\,dw}{\log(1+\beta w)} = \frac{2}{\beta^2}\,\frac{w'\,dw'}{\log(1+w')} = \frac{2}{\beta^2}\,\frac{(z-1)\,dz}{\log z}$$

$$= \frac{2}{\beta^2}\left[\left(\frac{z\,dz}{\log z}\right) - \frac{dz}{\log z}\right]$$

$$= \frac{2}{\beta^2}\left[\frac{d\,(z^2)}{\log z^2} - \frac{dz}{\log z}\right]$$

Now

$$\int \frac{dx}{\log x} = \log|\log x| + \log x + \frac{(\log x)^2}{2\,!\,2} + \frac{(\log x)^3}{3\,!\,3} + \dots$$

Hence, $h(t)$ is given implicitly by the relation

$$\begin{aligned}\log\left\{\frac{1+h}{1-h}\right\}^{1/2} &= \log|\log(1+\beta\sqrt{t})^2| - \log|\log(1+\beta\sqrt{t})| \\ &+ \log(1+\beta\sqrt{t})^2 - \log(1+\beta\sqrt{t}) \\ &+ \frac{1}{2\,!\,2}\left\{\left(\log(1+\beta\sqrt{t})^2\right)^2 - \left[\log(1+\beta\sqrt{t})\right]^2\right\} \\ &+ \dots\end{aligned}$$

or

$$\left(\frac{1+h}{1-h}\right)^{1/2} = \exp\left(f(x^2) - f(x)\right) \qquad f(x) = \int_1^x \frac{d\xi}{\log \xi}$$

where $x = 1 + \beta\sqrt{t}$.

Let

$$\frac{1+h}{1-h} = \exp\left(2\,[f(x^2) - f(x)]\right) = F(x)$$

one has

$$h = \frac{F - 1}{F + 1}$$

The function $F(x)$ and, hence, $h(x)$ may be evaluated simply by means of the digital computer. It should look like the curve shown in Fig. 9.

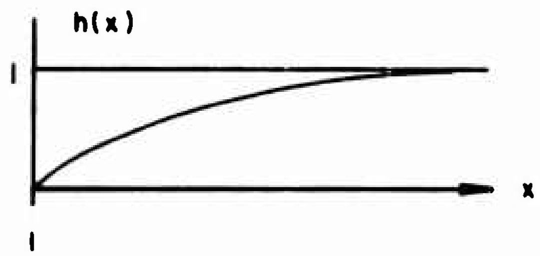

Figure 17

The function $h(x)$ is

$$h(x) = \frac{e^{g(x)} - 1}{e^{g(x)} + 1}$$

where

$$g(x) = \int_{x}^{x^2} \frac{d\xi}{\log \xi}$$

This is the same as

$$g(x) = \int_{\log x}^{\log x^2} \frac{e^y dy}{y} = \mathrm{Ei}\,(\log x^2) - \mathrm{Ei}\,(\log x)$$

$$= \mathrm{Ei}\,(2 \log x) - \mathrm{Ei}\,(\log x)$$

$$= \mathrm{Ei}(2\ell) - \mathrm{Ei}\,(\ell)$$

where $\ell = \log x$

Thus,

$$h(\log x) = h(\ell) = \frac{e^{g(\ell)} - 1}{e^{g(\ell)} + 1} = \tanh \left[\frac{1}{2}\left(\mathrm{Ei}\,(2\ell) - \mathrm{Ei}\,(\ell)\right)\right]$$

$$\ell = \log\,(1 + \beta\sqrt{t})$$

$h(\ell)$ is evaluated in the range $0 \leq \ell \leq 2$ and this is shown graphically in Fig. 10. It can be seen that this last method gives a filling time which depends only on the quantity R^2/ν. The accuracy of the result depends on the formula $r = R + 1.5\sqrt{\nu t}$.

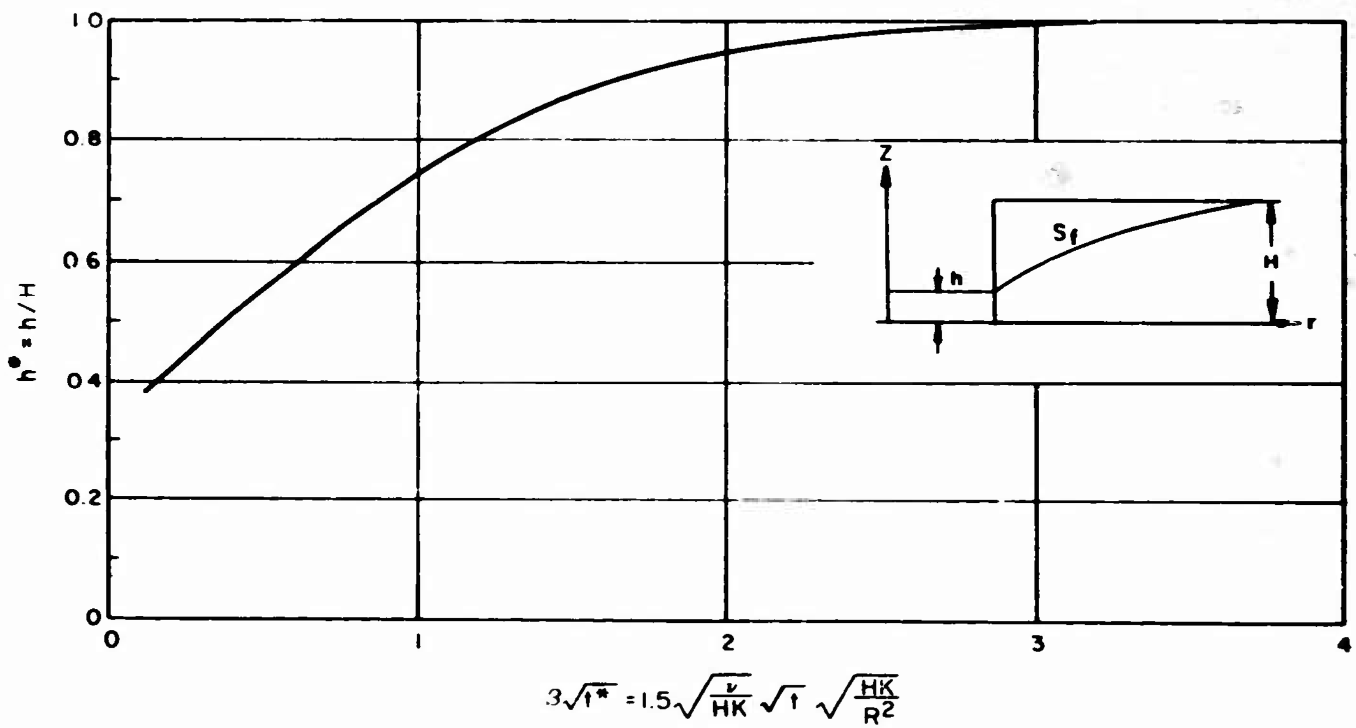

Figure 18

$$h^*(\beta\sqrt{t^*}) = \tanh\left\{\frac{1}{2}\left[Ei\left(2\log(1+\beta\sqrt{t^*})\right) - Ei\left(\log(1+\beta\sqrt{t^*})\right)\right]\right\}$$

PA-3-9273

3. THE FILLING OF A CRATER NEAR A RIVER IN AN UNCONFINED AQUIFER

The problem of the flow of ground water into an empty cylindrical crater in a medium with infinite extent has been considered in Section 2 and is relatively simple, due to circular symmetry. In the case of a crater located near a river or a lake, such as Detroit, the problem becomes rather complicated due to its nonsymmetry. An exact solution involves a considerable amount of computer calculation which is beyond the time limit. In order to obtain the numerical results, approximate solutions are given. These approximate solutions are based on some assumptions, in particular the Dupuit-Forscheimer assumption, as has been discussed in the previous sections.

Figure 19 indicates the general configuration of the problem. The elevation of the water along the river is assumed to be constant. That is to say the fluctuations of water level of the river are neglected. The mathematical formulation of the problem is similar to the problem indicated in Section 2 except along the river where $\varphi = k(p/r + z)$ is constant. The differential equation and boundary conditions can be written as follows:

The differential equation

$$\nabla^2\varphi = 0$$

The boundary conditions at the free surface are the same as in the case of an infinite aquifer, that is

a) Kinematic

$$m\frac{\partial h}{\partial t} + \frac{\partial \varphi}{\partial r}\frac{\partial h}{\partial r} - \frac{\partial \varphi}{\partial z} = 0$$

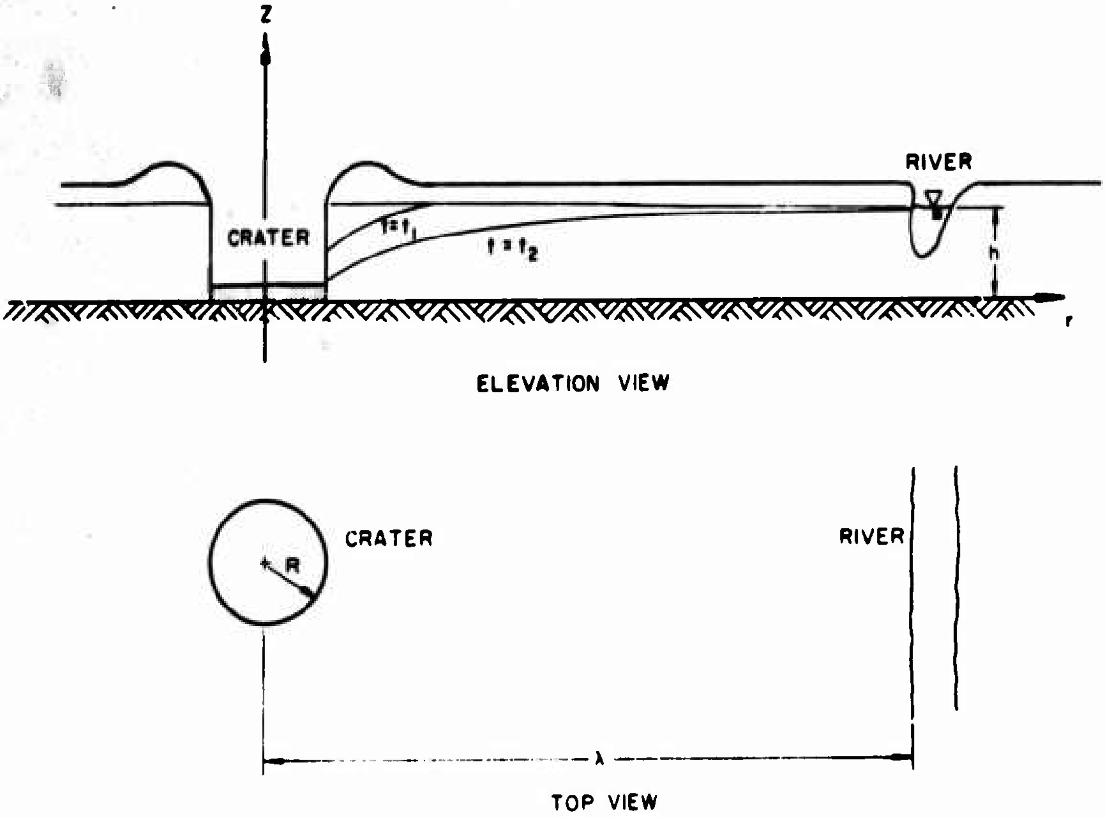

Figure 19

PA-3-9075

b) Dynamic

$$\varphi = - kz$$

The boundary condition along the river is $\varphi = - kH$.

The notations are as defined in the previous sections except λ which is the distance between the crater and the river.

One of the methods of solution to this problem is to use the method of images. On the opposite side of the river, as shown in Fig. 20, one introduces a recharge crater, with the recharge strength equal to the flow strength of the crater. One obtains the solution for each individual crater by use of power series as has been done in the previous section.

Since the equation of the velocity potential for different order is linear, we may superpose the solution of two individual craters and obtain the resultant solution which satisfies the boundary condition along the river. As has been demonstrated in the previous sections, the calculation of the numerical solution of one crater by use of power series expansion is quite laborious. Therefore, in the following section, an approximate solution is given from which numerical results may be calculated without too much work.

It is clear that the solution of this problem can be divided into two stages. Stage 1, when the radius of influence zone, r, is less than the distance between crater and river, λ. In this stage, the problem can be considered as if the river were not present. The solution has been obtained in the previous section. Stage 2, when the radius of influence zone, r, is greater than λ. In this stage, the river begins to effect the motion of the ground water. However, in this stage the gradient of the velocity potential φ becomes smaller (the gradient of velocity potential is maximum at initial instants and near the crater). The assumption of small drawdown may be used.

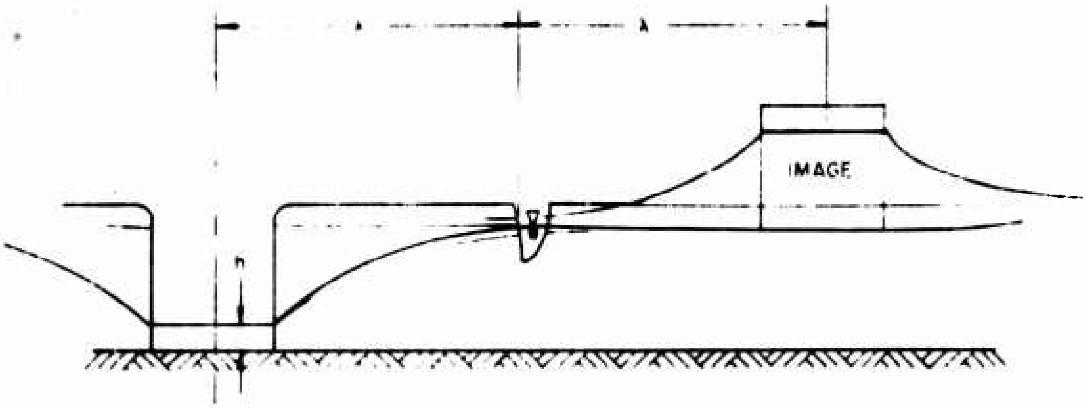

Figure 20

PA-3-9074

As the drawdown becomes smaller the discharge in a column of unit width, $\bar{q}$, may be approximated as

$$\bar{q} = -k\bar{H}\nabla h$$

where $\bar{H}$ is average depth of the aquifer. Substituting this equation into the unsteady continuity equation

$$\nabla \cdot \bar{q} = \epsilon \frac{\partial h}{\partial t}$$

where ϵ is the porosity gives

$$\nabla^2 h = \frac{\epsilon}{k\bar{H}} \frac{\partial h}{\partial t}$$

This is the usual heat equation. Since the equation is linear, one could also use the superposition method to satisfy the boundary condition along the river.

Writing the Laplace equation in cylindrical coordinates one has

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{\epsilon}{K\bar{H}}\frac{\partial h}{\partial t}$$

The solution to this equation with the boundary and initial conditions prescribed in Section 2 is

$$h = H_o - \frac{Q}{4\pi K\bar{H}} \int_{\frac{r^2 \epsilon}{4\pi\bar{H}t}}^{\infty} \frac{e^{-u}}{u} \, du$$

$$= H_o - \frac{Q}{4\pi K\bar{H}} \left[- E_i(-x) \right]$$

where

$$x = \frac{r^2 \epsilon}{4K\overline{H}t}$$

The drawdown is

$$H_o - h = \frac{Q}{4\pi K\overline{H}} \left[- E_i (-x) \right]$$

Superposing two craters in space as indicated in Fig. 21 one has

$$H_o - h = \frac{Q}{4\pi K\overline{H}} \left[E_i \left(- \frac{r_1^2 \epsilon}{4K\overline{H}t} \right) + E_i \left(- \frac{r_2^2 \epsilon}{4K\overline{H}t} \right) \right]$$

where r_1 and r_2 are distances from crater and image crater, respectively. They are equal to

$$r_1^2 = (x - \lambda)^2 + y^2$$

$$r_2^2 = (x + \lambda)^2 + y^2$$

in a rectangular system.

The equation governing the filling of the crater is

$$Q = \pi R^2 \frac{dh}{dt}$$

Substituting into the previous equation we have

$$\frac{dh}{dt} = \frac{4 \overline{H} K (H_o - h)}{R^2 \left[E_i \left(- \frac{r_1^2 \epsilon}{4K\overline{H}t} \right) + E_i \left(- \frac{r_2^2 \epsilon}{4K\overline{H}t} \right) \right]}$$

After integrating, we obtain

$$\log (H_o - h) = \frac{4\,\overline{H}\,K\,t}{R^2\left[E_i\left(-\frac{r_1^2\,\epsilon}{4K\overline{H}t}\right) + E_i\left(-\frac{r_2^2\,\epsilon}{4K\overline{H}t}\right)\right]} + c$$

Using the initial condition $t = 0$, $h = 0$, we have $c = \log H_o$

therefore

$$\frac{h}{H_o} = 1 - \exp \frac{4\,\overline{H}\,K\,t}{R^2\left[E_i\left(-\frac{r_1^2\,\epsilon}{4K\overline{H}t}\right) + E_i\left(-\frac{r_2^2\,\epsilon}{4K\overline{H}t}\right)\right]}$$

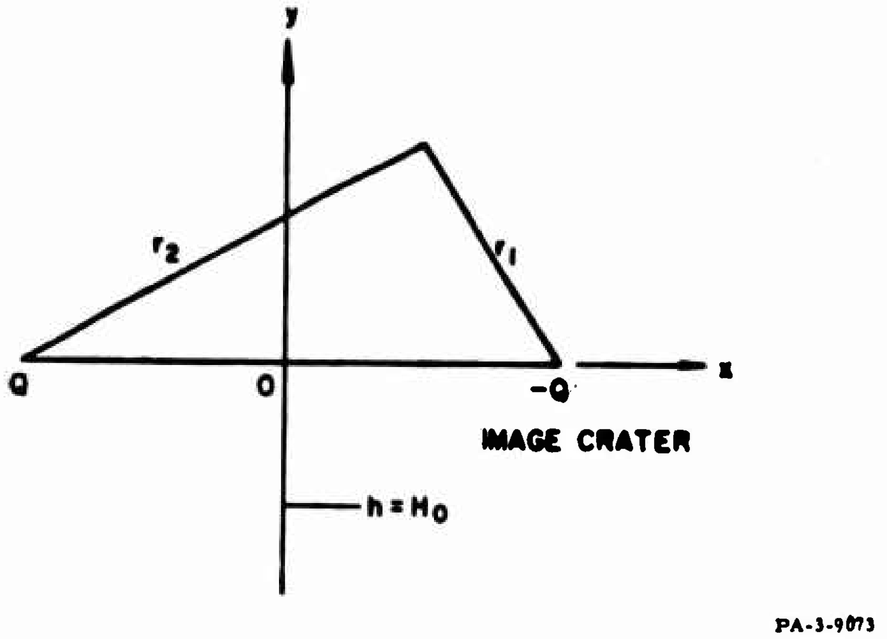

Figure 21

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4. DISCHARGE OF STREAM INTO A LARGE WELL IN A CONFINED AQUIFER

In the case where the explosion has created a cavity near a stream and the aquifer under consideration is confined, the solution of the problem of filling the cavity may be rather simply found with the notation shown in Fig. 22.

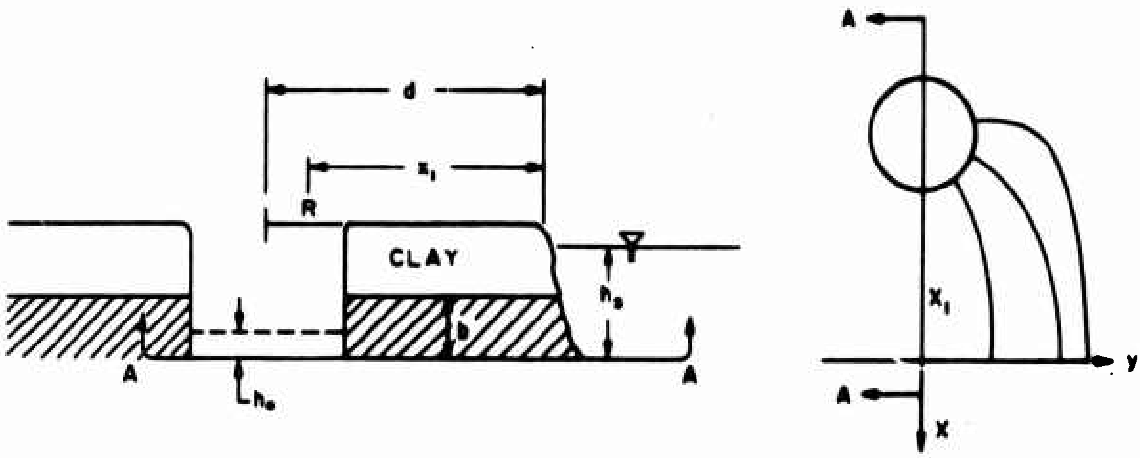

Figure 22

where

- b = the thickness of the aquifer
- x_1 = distance from origin to center of point well
- h_o = water level in well
- h_s = water level in stream
- d = distance from center of real well to stream
- R = radius of well
- x = coordinate
- y = coordinate

The solution may be written as

$$h_s - h = \frac{Q}{4\pi T} \ln \frac{(x - x_1)^2 + y^2}{(x + x_1)^2 + y^2}$$

where

Q = discharge

T = hydraulic transmissivity = Kb

K = permeability

This is the solution for a point well at $x = x_1$, $y = 0$ with a stream of head h_s along $x = 0$. The distance d and the well radius R are as yet unspecified.

For a finite diameter well, any equipotential may be taken as a well cross section. Thus, if h_w is the level of water in the well, then along the well edges

$$h_s - h_w = \frac{Q}{4\pi T} \ln \frac{(x - x_1)^2 + y^2}{(x + x_1)^2 + y^2}$$

and

$$\frac{(x - x_1)^2 + y^2}{(x + x_1)^2 + y^2} = \exp\left[\frac{4\pi T}{Q}(h_s - h_w)\right] = \alpha$$

This is the equation of a circle (the well edge).

$$(x - x_1)^2 + y^2 = \alpha (x + x_1)^2 + \alpha y^2$$

or

$$x^2 - \frac{1 + \alpha}{1 - \alpha} 2x_1 x + x_1^2 + y^2 = 0$$

$$x^2 - 2\left(x_1 \frac{1 + \alpha}{1 - \alpha}\right) x + \left(x_1 \frac{1 + \alpha}{1 - \alpha}\right)^2 + x_1^2 - \left(x_1 \frac{1 + \alpha}{1 - \alpha}\right)^2 + y^2 = 0$$

or

$$\left(x - x_1 \frac{1 + \alpha}{1 - \alpha}\right)^2 + y^2 = x_1^2 \left(\frac{1 + \alpha}{1 - \alpha}\right)^2 - x_1^2$$

Thus, the center is at $x = x_1 \frac{1 + \alpha}{1 - \alpha}$, and the radius is $x_1 \sqrt{\left(\frac{1 + \alpha}{1 - \alpha}\right)^2 - 1} = R.$

This solution may be easily adapted to a finite circular well at a distance d from the stream. The center is at

$$x_1 \frac{1 + \alpha}{1 - \alpha} \Rightarrow - x_1 \frac{1 + \alpha}{\alpha - 1}$$

since $\alpha > 1$, always, for a flow into well.

$$\therefore \quad d = -x_1 \frac{1+\alpha}{\alpha - 1} \qquad \text{or} \qquad -x_1 = \frac{\alpha - 1}{\alpha + 1} d$$

and

$$R = x_1 \sqrt{\left(\frac{1+\alpha}{1-\alpha}\right)^2 - 1} = \frac{\alpha - 1}{\alpha + 1}\sqrt{\left(\frac{\alpha + 1}{\alpha - 1}\right)^2 - 1}\ d$$

Recall that

$$\alpha = \exp\left[\frac{4\pi T}{Q}(h_s - h_w)\right] = e^{2\beta}$$

where

$$\beta = \frac{2\pi T}{Q}(h_s - h_w)$$

Thus,

$$\frac{\alpha - 1}{\alpha + 1} = \frac{e^{2\beta} - 1}{e^{2\beta} + 1} = \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}} = \tanh \beta$$

and

$$\frac{\alpha + 1}{\alpha - 1} = \coth \beta$$

Hence,

$$\left(\frac{\alpha + 1}{\alpha - 1}\right)^2 = \coth^2 \beta$$

thus,

$$\left(\frac{\alpha + 1}{\alpha - 1}\right)^2 - 1 = \coth^2 \beta - 1 = \operatorname{cosech}^2 \beta$$

$$\therefore \quad -x_1 = d \tanh \beta \qquad\qquad d = -x_1 \coth \beta$$

$$R = (\tanh \beta)(\operatorname{cosech} \beta)\, d = \frac{\sinh \beta}{\cosh \beta} \frac{1}{\sinh \beta}$$

$$= d \operatorname{sech} \beta = R$$

The procedure for obtaining the solution is (Fig. 23):

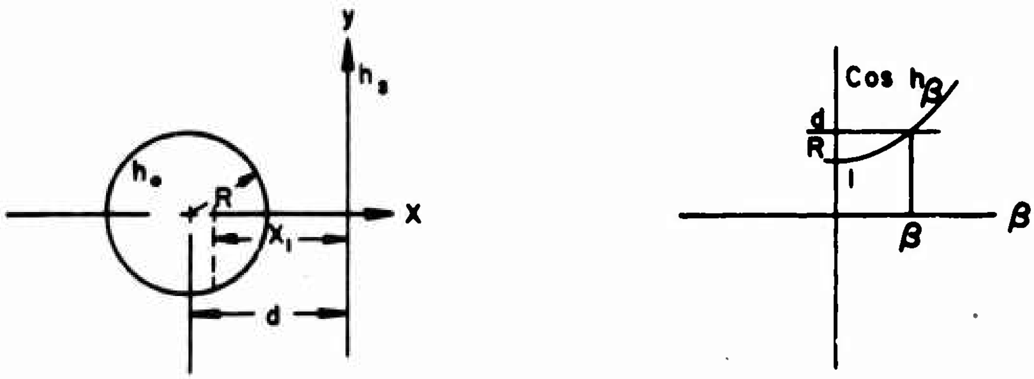

Figure 23

Given R, d, h_s, h_o, and K, b, etc.

1) Obtain β from $d = (\cosh \beta) R$

$d > R \qquad \frac{d}{R} = \cosh \beta$

2) Obtain x_1 from $|x_1| = d \tanh \beta$

3) $Q = 2\pi T/\beta \ (h_s - h_w)$

Solution:

$$\frac{4\pi T}{Q} (h_s - h) = \ln \frac{(x - d \tanh \beta)^2 + y^2}{(x + d \tanh \beta)^2 + y^2}$$

which gives equipotential lines and, by implication, flow lines.

To get filling times, etc., $h_w = h_w(t)$

$$Q(t) = \frac{2\pi T}{\cosh^{-1}\frac{d}{R}}(h_s - h_w(t)) = \frac{2\pi T\, h_s}{\cosh^{-1}\frac{d}{R}}\left[1 - \zeta(t)\right]$$

$$\zeta(t) = \frac{h_w(t)}{h_s}$$

One either has

$$\pi R^2 h_w(t) = \int_0^t Q(\tau)\, d\tau$$

which gives

$$\pi R^2 h_s\, \zeta(t) = \int_0^t Q(\tau)\, d\tau$$

or

$$\frac{d\zeta}{dt} = \frac{2T}{R^2 \cosh^{-1}\frac{d}{R}}\left[1 - \zeta\right] = C(1 - \zeta)$$

$$C = \text{constant} = \frac{2T}{R^2 \cosh^{-1}\frac{d}{R}}$$

$$\therefore \qquad \frac{d\zeta}{1 - \zeta} = C\, dt$$

Integrating

$$-\log(1 - \zeta) = C\, t + \text{constant}$$

The condition $t = 0$, $\zeta = 0$ implies that the constant = 0

$$\therefore \qquad 1 - \zeta = \exp\left(-\frac{2T\, t}{R^2 \cosh^{-1}\frac{d}{R}}\right)$$

or

$$\frac{h_w(t)}{h_s} = 1 - \exp\left(- \frac{2T\,t}{R^2 \cosh^{-1} \frac{d}{R}}\right)$$

and finally,

$$h_w(t) = h_s\left[1 - \exp\left(- \frac{2T\,t}{R^2 \cosh^{-1} \frac{d}{R}}\right)\right]$$

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5. A QUIESCENT WELL IN A UNIFORM FLOW

If, after the crater is filled, one can assume there is a uniform flow over the well as illustrated in Fig. 24, some fluid will pass through the well, and some fluid will bypass it. If the fluid in the well were radioactive, and if it were assumed that the radioactivity is simply advected, it would be of interest to know the width of the zone of contamination downstream.

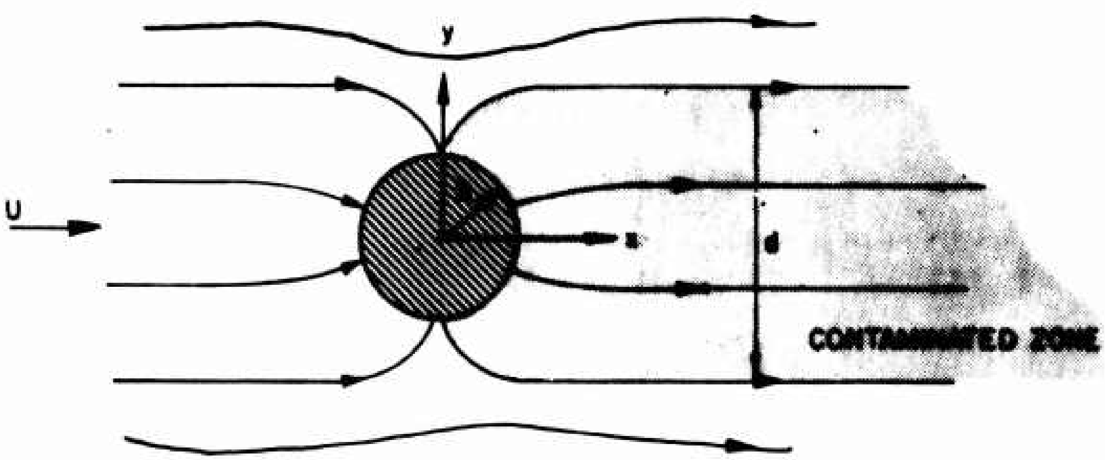

Figure 24

It will be shown in this section that $d = 4R$.

With the x, y coordinate system shown in Fig. 24, the mathematical problem is

$$\nabla^2 h = 0 \text{ for } r = \sqrt{x^2 + y^2} > R$$

$$-k\frac{\partial h}{\partial x} = U \text{ for } r \to \infty$$

$$h = \text{constant on } r = R$$

Without loss of generality, this constant can be taken to be zero. The solution to this problem can be found easily, and it is

$$h = -\frac{U}{k}\cos\theta\left(r - \frac{R^2}{r}\right)$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} y/x$$

This is the real part of

$$-\frac{U}{k}\left[z - \frac{R^2}{z}\right] \qquad z = x + iy$$

The imaginary part is

$$\omega = -\frac{U}{k}\left(y + \frac{R^2 y}{x^2 + y^2}\right)$$

The value of ω at $y = R$, $x = 0$ is

$$-\frac{U}{k} 2R$$

Thus, the critical streamline (Fig. 25) is represented by the equation

$$\frac{U}{k} 2R = \frac{U}{k}\left[y + \frac{R^2 y}{x^2 + y^2}\right]$$

as $x \to \infty$, this implies $y = 2R = d/2$. Thus, the thickness of the zone of contamination d is $4R$.

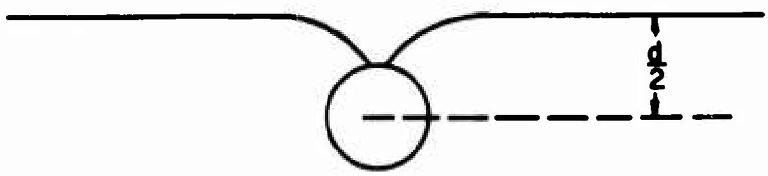

Figure 25

The foregoing analysis is based on nondispersive flow. Actually, the radioactivity would, of course, disperse transversely from the zone of contamination with the result of a larger zone than 4R.

To investigate dispersion in a porous medium, the situation could be idealized and the case of a simple mixing zone (Fig. 26) is investigated.

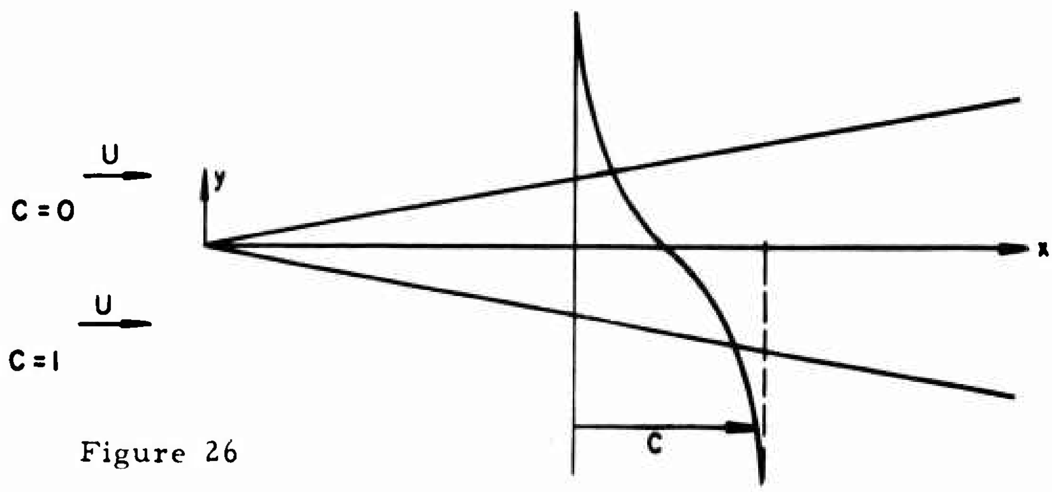

Figure 26

The only variable is the concentration C, and the governing equation is the dispersion equation

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = \frac{\partial}{\partial x}\left(D_x \frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_y \frac{\partial C}{\partial y}\right)$$

which, in the present case, since $v = 0$ and $U = u$, becomes

$$U \frac{\partial C}{\partial x} = \frac{\partial}{\partial x}\left[D_x \frac{\partial C}{\partial x}\right] + \frac{\partial}{\partial y}\left[D_y \frac{\partial C}{\partial y}\right]$$

Here D_x and D_y are the dispersion coefficients in the longitudinal and transverse directions.

Neglecting the D_x term compared with the D_y term on the ground of a boundary layer (BL) assumption, we have

$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial y}\left(\epsilon \frac{\partial C}{\partial y}\right)$$

where $\epsilon = D_y/U$. Since U is constant, D_y would be expected to be constant also. Hence, ϵ is constant. Absorb ϵ in y to obtain

$$\frac{\partial C}{\partial x} = \frac{\partial^2 C}{\partial y^2}$$

BC $\quad y \to \infty, \quad C = 0$

$\qquad\qquad\qquad\qquad x > 0$

$\quad y \to -\infty, \quad C = 1$

$\quad y \to \infty, \quad \partial C/\partial y \to 0$

For a similarity solution, let

$$\eta = (y/\sqrt{x})$$

and $c = f(\eta)$; then

$$\frac{\partial C}{\partial x} = -\frac{1}{2}\frac{y}{x^{3/2}} f'$$

$$\frac{\partial^2 C}{\partial y^2} = f'' \frac{1}{x}$$

Thus, DE becomes

$$\frac{1}{2}\eta f' + f'' = 0$$

and BC becomes

$$f(\infty) = 0 \quad f(-\infty) = 1$$

The solution is $\quad C = -1/2\sqrt{\pi}$

$$\therefore \quad f(\eta) = +\frac{1}{\sqrt{\pi}} \int_{\eta/2}^{\infty} e^{-z^2}\, dz$$

This is the standard diffusion type solution and may be found in textbooks. The value of D_y needs to be estimated. In Ref. 2 a graph shows D_y/D as a function of A_pV/D where

A_p = particle radius of the bed material

V = velocity (U in this case)

D = diffusion coefficient (molecular)

For large velocities V, (or for $A_pV/D \sim 10$) D_y/D is about 5 to 10. Hence, D_y can be taken to be about D to $10D$ for practical purposes.

The similarity variable is

$$\eta/2 = \frac{1}{2}\frac{y}{\sqrt{x}\sqrt{D_y/U}} = \frac{y}{2\sqrt{D_y x/U}}$$

Let y_{90} be the y where $C = 1/10$.

This means

$$f = 1/10 \Rightarrow \frac{y_{90}}{2\sqrt{(D_y/U)\,x}} \approx 1.64$$

Thus

$$y_{90} \quad 3\sqrt{D_y/U}\,\sqrt{x}$$

The mixing zone, therefore, grows like $\sqrt{x}$ with a proportionality constant

$$\approx 3\sqrt{10D_y/U}$$

Applied to the present problem, the flow situation would appear as shown in Fig. 27.

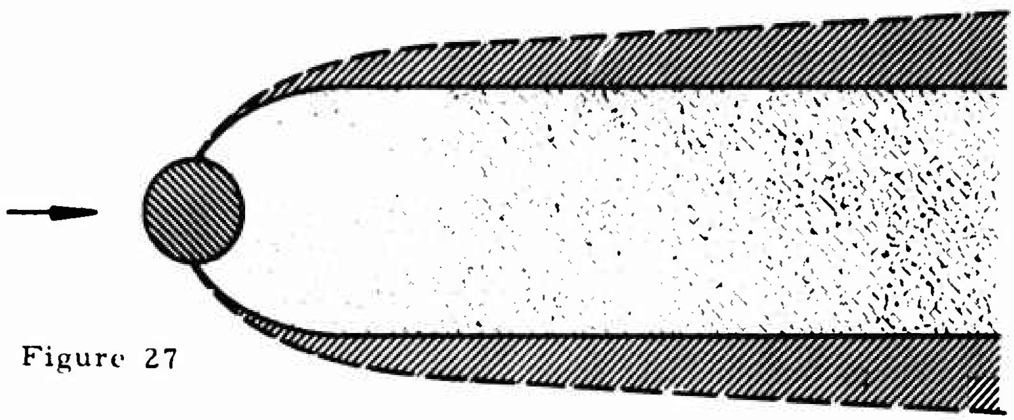

Figure 27

Since the substance undergoing dispersion is radioactive in nature, there is a natural rate of decay. Thus, the equation for dispersion should be modified by a decay term. For those isotopes that decay very slowly in comparison with the motion in the porous medium, the above nondecay analysis should apply. For those isotopes which decay reasonably rapidly, the analysis of the problem should be as shown in Fig. 28.

Still considering the simple case of a mixing zone,

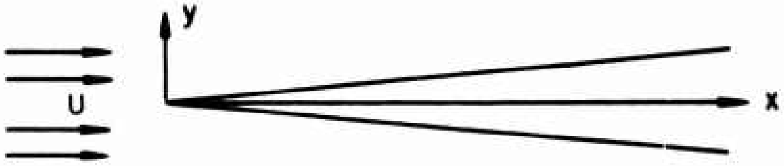

Figure 28

the governing equation is

$$U \frac{\partial C}{\partial x} + kC = D_y \frac{\partial^2 C}{\partial y^2}$$

where it is assumed that the radioactive decay rate is proportional to the concentration with k as the proportionality constant. Dividing by U,

$$\frac{\partial C}{\partial x} + \frac{k}{U} C = \epsilon \frac{\partial^2 C}{\partial y^2}$$

The special case $\epsilon = 0$ gives the solution

$$C = C_o \exp\left(- \frac{k}{U} x\right)$$

where C_o is the concentration at $x = 0$. This may be applied to the

core of the zone of contamination in the previous problem. This may now be taken as the boundary condition at $y \to -\infty$.

Nondimensionalizing the variables by

$$C^* = C/C_o, \quad x^* = (k/U)\, x, \quad y^* - \sqrt{(k/D_y)}\; y$$

the following equation is obtained

$$\frac{\partial C^*}{\partial x^*} + C^* = \frac{\partial^2 C^*}{\partial y^{*2}}$$

with the boundary conditions

$$C^* \to 0 \text{ as } y^* \to \infty, \text{ for } x^* > 0$$

$$C^* \to e^{-x^*} \text{ as } y^* \to - \quad , \quad x^* \to 0$$

$$C^* \to 0 \text{ as } x^* \to$$

Let

$$C^* = e^{-x^*} f$$

$$\frac{\partial C^*}{\partial x^*} = -e^{-x^*} f + e^{-x^*} f_{x^*}$$

$$\frac{\partial^2 C^*}{\partial y^{*2}} = e^{-x^*} f_{y^* y^*}$$

$\therefore$ the equation becomes

$$\frac{\partial f}{\partial x^*} - f + f = \frac{\partial^2 f}{\partial y^{*2}}$$

or

$$\frac{\partial f}{\partial x^*} = \frac{\partial^2 f}{\partial y^{*2}}$$

The boundary conditions become

$$\left.\begin{array}{l} f \to 0 \text{ as } y^* \to \infty \\ f \to 1 \text{ as } y^* \to -\infty \end{array}\right\} x > 0$$

Now note that this is the same problem as the case of nondecay dispersion. Hence, the solution is

$$f = \frac{1}{\sqrt{\pi}} \int_{\eta/2}^{\infty} e^{-\xi^2} d\xi$$

Hence,

$$C = C_o \exp\left(-\frac{k}{U} x\right) \frac{1}{\sqrt{\pi}} \int_{\eta/2}^{\infty} e^{-\xi^2} d\xi \quad //$$

6. FLOW INTO A WATER SURROUNDED CRATER

In the case of the New Orleans burst, flooding could extend to areas of considerable size, and the water will certainly surround the crater lip. Therefore, the time of filling of the crater would be different from the case of infinite media.

If the Dupuit-Forchheimer assumption is used and initial transient phenomena are neglected, we immediately have the equation for groundwater motion

$$H^2 - h^2 = \frac{Q}{\pi K} \log \frac{r_o}{R}$$

The notations are indicated in Fig. 29, except Q and K which are defined in the previous sections to be discharge and permeability.

Another equation which governs the filling of the crater is

$$\int_0^t Q(\tau)\, d\tau = \pi R^2 h$$

or

$$Q = \pi R^2 \frac{dh}{dt}$$

Combining the equation with the equation of motion, we have

$$\pi R^2 \frac{dh}{dt} = \frac{\left(H^2 - h^2\right) \pi k}{\log \frac{r_o}{R}}$$

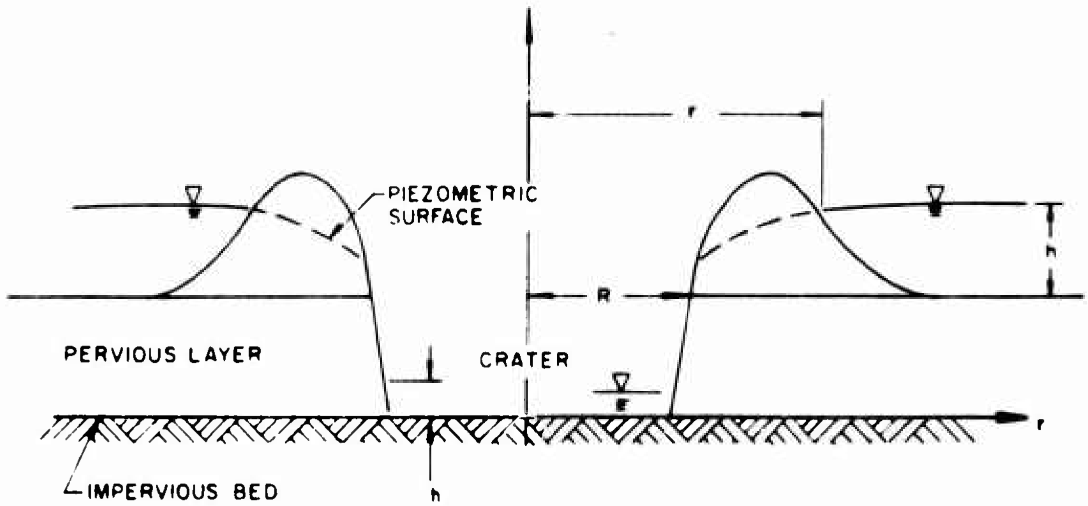

Figure 29

PA-3-9072

or

$$\frac{dh}{H^2 - h^2} = \frac{K\,dt}{R^2 \log \frac{r_o}{R}}$$

After integrating, we obtain

$$\frac{1}{H} \tanh^{-1} \frac{h}{H} = \frac{Kt}{R^2 \log \frac{r_o}{R}} + c$$

The initial conditon, $h = 0$ when $t = 0$, implies $c = 0$.

Therefore, one has

$$h = H \tanh \frac{H\,Kt}{R^2 \log \frac{r_o}{R}}$$

7. REFERENCES

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13. ABSTRACT

This fifth and final volume of the report, "Hydrodynamic Effects of Nuclear Explosions," presents new theoretical developments of two problems. The first is the determination of the waves resulting from the passage of a high-pressure disturbance over the free surface of a body of water. This would occur in the case of a burst on or over land near a shore and is, therefore, of interest to the Five City Study in which three nuclear surface bursts are near rivers or bays.

The second topic is motion of the ground water table induced by a surface burst. This problem is of interest for the determination of the migration of radioactive contaminants and is also applicable to three cities of the Five City Study.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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